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Inconsistency-tolerant reasoning with OWL DL

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ABSTRACT

The Web Ontology Language (OWL) is a family of description logic based ontology languages for the Semantic Web and gives well defined meaning to web accessible information and services. The study of inconsistency-tolerant reasoning with description logic knowledge bases is especially important for the Semantic Web since knowledge is not always perfect within it. An important challenge is strengthening the inference power of inconsistency-tolerant reasoning because it is normally impossible for paraconsistent logics to obey all important properties of inference together. This paper presents a non-classical DL called quasi-classical description logic (QCDL) to tolerate inconsistency in OWL DL which is a most important sublanguage of OWL supporting those users who want the maximum expressiveness while retaining computational completeness (i.e., all conclusions are guaranteed to be computable) and decidability (i.e., all computations terminate in finite time). Instead of blocking those inference rules, we validate them conditionally and partially, under which more useful information can still be inferred when inconsistency occurs. This new non-classical DL possesses several important properties as well as its paraconsistency in DL, but it does not bring any extra complexity in worst case. Finally, a transformation-based algorithm is proposed to reduce reasoning problems in QCDL to those in DL so that existing OWL DL reasoners can be used to implement inconsistency-tolerant reasoning. Based on this algorithm, a prototype OWL DL paraconsistent reasoner called PROSE is implemented. Preliminary experiments show that PROSE produces more intuitive results for inconsistent knowledge bases than other systems in general.

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1. Introduction

As an extension of the World Wide Web (WWW), the Semantic Web [3] becomes more constantly changing and highly collaborative. Ontologies considered one of the pillars of the Semantic Web will rarely be perfect due to many reasons, such as modeling errors, migration from other formalisms, merging ontologies, and ontology evolution [35,13,32,11,28,5]. As a fragment of predicate logic [8], description logic (DL), which is the logical foundation of the Web Ontology Language [26] (e.g., sublanguages OWL Lite and OWL DL correspond to $\mathcal{SHIF}(\mathbf{D})$ and $\mathcal{SHOIN}(\mathbf{D})$ respectively), is unable to tolerate inconsistencies occurring in knowledge bases (KBs). Thus, the topic of inconsistency handling in OWL and DL has received extensive interests in the community in recent years [35,32,24].

There are several approaches to handling inconsistencies in DLs. All of them can be functionally roughly classified into two different types. One type is based on the assumption that inconsistencies indicate erroneous data which are to be removed in order to obtain a consistent knowledge base (KB) [35,13,18,33,15,22,10]. In these approaches, researchers hold a

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common view that KBs should be completely free of inconsistencies, and thus try to eliminate inconsistencies from them to recovery consistency immediately by any means possible. However, there are some different opinions about the first type of treating inconsistency. For instance, [4] regarded the first type as "*too simplistic for developing robust intelligent systems, and fails to use the benefits of inconsistent knowledge in intelligent activities, or to acknowledge the fact that living with inconsistencies seems to be unavoidable*". And [4] argues that inconsistencies in knowledge are the norm in the real world, and so should be formalized and used, rather than always rejected. The other, called inconsistency-tolerant (or paraconsistent) approaches, is to not simply avoid inconsistencies but apply non-standard reasoning methods (e.g., non-standard inference or non-classical semantics) to obtain meaningful answers [38,23,30,12,45,20,29,24,43]. In the second type of approaches, inconsistency treated as a natural phenomenon in realistic data, should be tolerated in reasoning. So far, the main idea of existing paraconsistent methods for handling inconsistency is introducing either non-standard inference or non-classical semantics to draw meaning conclusions from inconsistent KBs [38]. Those paraconsistent approaches with non-standard inference presented by [43,12] are employing argument principles where consistent subsets are selected from an inconsistent KB as substitutes in reasoning. Those paraconsistent approaches are based on multi-valued semantics (a popular kind of non-classical semantics) such as four-valued DL studied by [38,23,30,24] based on Belnap's four-valued semantics [2], paradoxical DL presented by [45] based on Priest's paradoxical semantics, three-valued DL discussed by [29] based on Kleene's three-valued semantics, and [20] based on a dual interpretation semantics.

Multi-valued logic, as a family of non-classical logics, are successful in handling inconsistency and uncertainty in DL such as four-valued DL [39,23,24] and fuzzy DLs [40,6,5,9]. Because four-valued logic is a basic member of the family of multi-valued logics, four-valued semantics of DL has got a lot of attention [38,23,24]. However, the inference power of four-valued DL is rather weak as noted/argued by [23,45,20,24] although three kinds of implications (namely, *material implication*, *internal implication* and *strong implication*, see Section 3) are introduced in four-valued DL to improve inference power. Some important properties (their formalizations can be found in Section 3.1) about inference such as *disjunctive syllogism*, *resolution* and *intuitive equivalence* are invalid in four-valued DL. For instance, assume that Wade is a student or a staff in a university and Wade is not a student. However, we do not conclude that Wade is a staff in that university since disjunctive syllogism fails in four-valued DL. And, though we know that all PhD students are students, we cannot infer that all persons are non-PhD students or students since intuitive equivalence also fails in four-valued DL. Moreover, there exist some quite differences among three implications. For instance, *modus ponens* is valid for internal implication and strong implication while it is invalid for material implication. *Modus tollens* is valid for strong implication while it is invalid for material implication and internal implication. Thus users have to make a suitable choice before reasoning. The weak inference power of four-valued DL is because a concept and its negation treated as two fully independent concepts.

We use two practical examples to show why the properties of disjunctive syllogism, modus ponens and modus tollens are useful in daily life ontology.

In wine ontology,[1] we have an axiom $WineDescriptor \equiv WineColor \sqcup WineTaste$. If we additionally know that both $WineDescriptor(Strong)$ and $\neg WineColor(Strong)$, we will expect that $Strong$ is a $WineTaste$, which can be inferred by the property of disjunctive syllogism.

In the Pizza ontology, we know that $IceCream \sqsubseteq \neg Pizza$. On the one hand, if we additionally know $IceCream(a)$, we will expect $\neg Pizza(a)$, which is captured by the property of modus ponens. In the other hand, if we additionally know $Pizza(a)$, we will expect $\neg IceCream(a)$, which is captured by the property of modus tollens. In addition, if we additionally know $IceCream(a)$ and $Pizza$, then we will expect the inconsistency about a to be tolerated in the reasoning.

Indeed, the weak inference power is one of common characteristics of the family of paraconsistent logics where some important inference rules are prohibited in order to avoid the explosion of inference [4]. As a result, this topic of making more properties about inference valid under preventing the explosion of inference becomes interesting and important since more useful information can be inferred from inconsistent KBs [30].

To avoid the shortcomings of four-valued DL, in this paper, we investigate the problem of defining a suitable non-classical DL based on the *quasi-classical logic* (QC logic) proposed by [19]. This problem is challenging in that it is not straightforward to extend the semantics of QC logic to DLs. Specifically, in the setting of DLs, it is difficult to define a suitable meaning for the two logical connectives "$\sqcup$" and "$\sqsubseteq$", which are two key constructors in DLs. To solve this problem, we define a new description logic called *quasi-classical description logic* (QCDL, for short), which allows to infer more useful information from inconsistent KBs. To achieve QCDL, we first extend the syntax of DLs by introducing a new connective called *QC negation* and then we introduce a new semantics for our proposal DL by two satisfactions, namely weak satisfaction and strong satisfaction. Informally, the former is to obtain a satisfactory paraconsistency while the latter is to make most properties of inference mentioned above (their formalization can be found in Section 2) valid. We show that QCDL can be applied to tolerate inconsistency in reasoning with DLs and our proposal approach can improve paraconsistent reasoning by using a different principle from four-valued DL. To employ off-the-shelf DL reasoners, we develop a transformation-based algorithm by reducing QCDL into classical DL and then implement a prototype reasoning system named PROSE. Our implementation can be seen as an extension of classical DL reasoners in this sense that PROSE can still infer meaningful information in an inconsistent ontology while classical DL reasoners will crash when there is an inconsistency. Finally, we show that the complexity of QC consistency problem in QCDL is not higher than that of consistency problem in DL. It should be noted

[1] http://www.w3.org/TR/owl-guide/wine.rdf.

that our technique can be feasibly extended in the whole DL family including $\mathcal{SROIQ}(\mathbf{D})$ (i.e., the logical foundation of OWL 2 [26]). In this paper, we still use $\mathcal{SHOIN}(\mathbf{D})$, as a logical foundation of OWL DL, to simplify our discussion and make readers understand our technique clear.

Compared with classical QC logic [19], a new connective defined in the syntax of QCDL called QC negation, is helpful for implementing paraconsistent reasoning in a intuitive way. Because of this, the language of DL is a sublanguage of QCDL. The semantics of QCDL can tolerate inconsistency occurring in DL KBs and two important reasoning tasks of QCDL can be reduced into the corresponding tasks of DL. Moreover, we define a suitable meaning for the two logical connectives "$\sqcup$" and "$\sqsubseteq$", which consist of two key constructors in DLs. In this sense, we generalize classical QC logic and discuss some new interesting reasoning tasks such as QC satisfiable problem and QC inconsistent problem in QCDL.

Our previous work [44] introduces primary quasi-classical semantics for $\mathcal{ALC}$, a simple member of description logics, and presents some properties of it and the relationship with the method by using four-valued logic. However, we investigated that there exist three insufficiencies as follows:

1. The approach presented in [44] is not enough to characterize all features of DLs. As we all known, DL is used in artificial intelligence (AI) for formal reasoning on the concepts of an application domain [1]. However, in our previous approach [44], we did not really introduce a new logic with quasi-classical negation, but rather viewed negation as a transformation on formulas (axioms). This makes it impossible to directly represent the "opposite" concept of a given concept, because the negation of a concept $\neg C$ is not taken as the "opposite" concept but rather as a concept unrelated to C. In this paper, we can directly introduce the QC negation of a concept as the "opposite" concept. Thus, we can further discuss QC satisfiable concepts. Moreover, our previous approach cannot capture the natural relationship between "$\sqcup$" and "$\sqsubseteq$".
2. The basic approach of [44] cannot be generalized to more expressive description logics such as $\mathcal{SROIQ}(\mathbf{D})$ (i.e., the logical foundation of OWL 2 [26]) which can be obtained by generalizing the present paper. One of important reasons is the complement of axioms $\sim\phi$ presented in [44] cannot capture expressive DL axioms. For instance, $\sim(\geqslant n\,R.C)(a)$ cannot be represented by both $\leqslant(n-1)R(a)$ and $\sim C(b)$. Instead, the QC negation of concept $\bar{C}$ introduced in this paper can capture expressive DL axioms, e.g., $\overline{\geqslant n\,R.C}(a) \equiv \leqslant(n-1)\,R.\bar{C}(a)$. We investigated that the QC negation can also capture all DL axioms even in $\mathcal{SROIQ}(\mathbf{D})$.
3. The complement of inclusions $\sim(C \sqsubseteq D)$ can be no longer translated into a corresponding DL concept inclusions. Because of this, it is impossible to transform this logic into classical DL. On the contrary, our proposal QCDL can be exactly transformed into DL.

In short, this paper does not merely attempt to give a QC semantics to a standard description logic, but instead we define a QC description logic (called QCDL).

This paper extends our previous DL-2009 and ESWC-2009 papers by developing a transformation-based algorithm and implementing it as a paraconsistent prototype reasoner. In addition, we analyze and evaluate some experimental results.

The rest of this paper is organized as follows: Section 2 reviews briefly DLs and paraconsistent logics. Section 3 introduces the syntax and semantics of QCDL and applies QCDL in paraconsistent reasoning with DL. Section 4 develops a transformation-based algorithm. Section 5 presents our PROSE and some evaluation results. Section 6 compares QCDL with others paraconsistent DLs. Finally, Section 7 concludes the paper.

2. Preliminaries

In this section, we give a brief introduction of description logics and paraconsistent logic.

2.1. Description logics

In description logics (DLs), elementary descriptions are *concept names* (unary predicates) and *role names* (binary predicates). Complex descriptions are built from them inductively using concept and role constructors provided by the particular DLs under consideration. In this section, we review the syntax and semantics of DLs. For more comprehensive background knowledge of DLs, we refer the reader to some basic references [1,16].

Let N_C, N_R, and N_I be countably infinite sets of concept names, role names, and individual names. $N_R = \mathbf{R}_A \cup \mathbf{R}_D$ where $\mathbf{R}_A$ is a set of abstract role names and $\mathbf{R}_D$ is a set of concrete role names. The set of *roles* is then $N_R \cup \{R^- \mid R \in N_R\}$ where R^- is the inverse role of R. The function $Inv(\cdot)$ is defined on the sets of roles as follows, where R is a role name: $Inv(R) = R^-$ and $Inv(R^-) = R$. For roles R_1 and R_2, a *role axiom* is either a role inclusion, which is of the form $R_1 \sqsubseteq R_2$ for $R_1, R_2 \in \mathbf{R}_A$ or $R_1, R_2 \in \mathbf{R}_D$, or a transitivity axiom, which is of the form $\mathrm{Trans}(R)$ for $R \in \mathbf{R}_A$. A *role hierarchy* $\mathcal{R}$ (or an *RBox*) is a finite set of role axioms. Let $\sqsubseteq^*_{\mathcal{R}}$ be the reflexive–transitive closure of $\sqsubseteq$ on $\mathcal{R}$ as follows: $\{(R_1, R_2) \mid R_1 \sqsubseteq R_2 \in \mathcal{R}$ or $Inv(R_1) \sqsubseteq Inv(R_2) \in \mathcal{R}\}$. A role R is *transitive* in $\mathcal{R}$, if a role R' exists such that $R' \sqsubseteq^*_{\mathcal{R}} R$, $R \sqsubseteq^*_{\mathcal{R}} R'$, and either $\mathrm{Trans}(R') \in \mathcal{R}$ or $\mathrm{Trans}(Inv(R')) \in \mathcal{R}$. A role S is *simple* if no transitive role R exists such that $R \sqsubseteq^*_{\mathcal{R}} S$. R^{tc} denotes the transitive closure of R.

Concrete datatypes are used to represent literal values such as numbers and strings. A type system typically defines as a set of "primitive" datatypes, such as *string* or *integer*, and provides a mechanism for deriving new datatypes from existing

Table 1
Syntax and semantics of $\mathcal{SHOIN}(\mathbf{D})$.

Elements	Syntax	Semantics
individual (N_I)	a	$a^{\mathcal{I}_c} \in \Delta^{\mathcal{I}_c}$
atomic concept (N_C)	A	$A^{\mathcal{I}_c} \subseteq \Delta^{\mathcal{I}_c}$
abstract role ($\mathbf{R}_A$)	R	$R^{\mathcal{I}_c} \subseteq \Delta^{\mathcal{I}_c} \times \Delta^{\mathcal{I}_c}$
concrete role ($\mathbf{R}_D$)	T	$T^{\mathcal{I}_c} \subseteq \Delta^{\mathcal{I}_c} \times \Delta_{\mathbf{D}}^{\mathcal{I}_c}$
datatype ($\mathbf{D}$)	d	$d^{\mathbf{D}} \subseteq \Delta_{\mathbf{D}}$
inverse abstract role	$Inv(R)$	$\{(x, y) \mid (y, x) \in R^{\mathcal{I}_c}\}$
Complex concepts		
top concept	$\top$	$\top^{\mathcal{I}_c} = \Delta^{\mathcal{I}_c}$
bottom concept	$\bot$	$\bot^{\mathcal{I}_c} = \emptyset^{\mathcal{I}_c}$
negation	$\neg C$	$(\neg C)^{\mathcal{I}_c} = \Delta^{\mathcal{I}_c} \setminus C^{\mathcal{I}_c}$
conjunction	$C \sqcap D$	$(C \sqcap D)^{\mathcal{I}_c} = C^{\mathcal{I}_c} \cap D^{\mathcal{I}_c}$
disjunction	$C \sqcup D$	$(C \sqcup D)^{\mathcal{I}_c} = C^{\mathcal{I}_c} \cup D^{\mathcal{I}_c}$
exist restriction	$\exists R.C$	$\{x \mid \exists y.(x, y) \in R^{\mathcal{I}_c}$ and $y \in C^{\mathcal{I}_c}\}$
value restriction	$\forall R.C$	$\{x \mid \forall y.(x, y) \in R^{\mathcal{I}_c}$ implies $y \in C^{\mathcal{I}_c}\}$
nominal ($\mathcal{O}$)	$\{o\}$	$\{o\}^{\mathcal{I}_c} \subseteq \Delta^{\mathcal{I}_c}$, $\sharp(\{o\}^{\mathcal{I}_c}) = 1$
number restriction ($\mathcal{N}$)	$\geqslant nR$	$\{x \mid \sharp(\{y.(x, y) \in R^{\mathcal{I}_c}\}) \geqslant n\}$
	$\leqslant nR$	$\{x \mid \sharp(\{y.(x, y) \in R^{\mathcal{I}_c}\}) \leqslant n\}$
datatype exists	$\exists T.d$	$\{x \in \Delta^{\mathcal{I}_c} \mid \exists y.(x, y) \in T^{\mathcal{I}_c}$ and $y \in d^{\mathbf{D}}\}$
datatype value ($\mathbf{D}$)	$\forall T.d$	$\{x \in \Delta^{\mathcal{I}_c} \mid \forall y.(x, y) \in T^{\mathcal{I}_c}$ implies $y \in d^{\mathbf{D}}\}$

ones. To represent concepts such as "persons whose age is at least 21", a set of concrete datatypes $\mathbf{D}$ is given, and, with each $d \in \mathbf{D}$, a set $d^{\mathbf{D}} \subseteq \Delta_{\mathbf{D}}$ is associated, where $\Delta_{\mathbf{D}}$ is the domain of all datatypes. Assume that:

(1) the domain of interpretation of all concrete datatypes $\Delta_{\mathbf{D}}$ (the concrete domain) is disjoint from the domain of interpretation of our concept language (the *abstract domain*); and
(2) there exists a sound and complete decision procedure for the emptiness of an expression of the form $d_1^{\mathbf{D}} \cap \cdots \cap d_n^{\mathbf{D}}$, where d_i is a (possibly negated) concrete datatype from $\mathbf{D}$ (where $\neg d$ is interpreted as $\Delta_{\mathbf{D}} \setminus d^{\mathbf{D}}$).

A set of datatypes is *conforming* if it satisfies the above criteria. The set of concepts is the smallest set such that each concept name $A \in N_C$ is a concept, complex concept in $\mathcal{SHOIN}(\mathbf{D})$ are formed according to the following syntax rule by using the operators shown in Table 1:

$$C, D \to A \mid d \mid \top \mid \bot \mid \neg C \mid C \sqcup D \mid C \sqcap D \mid \exists R.C \mid \forall R.C \mid \{o\} \mid \geqslant nR \mid \leqslant nR \mid \exists T.d \mid \forall T.d; \tag{1}$$

where $o \in N_I$, C, D concepts, R an abstract role, T a concrete role, S a simple role and $d \in \mathbf{D}$ a concrete datatype.

Note that the disjunction of nominals $\{o_1\} \sqcup \cdots \sqcup \{o_m\}$, where o_i $(1 \leqslant i \leqslant m)$ and m is a positive integer, is still taken as a nominal, denoted by $\{o_1, \ldots, o_m\}$. Indeed, nominals can be technically treated as complex concepts.

In this paper, let A, B (or with A_i, B_i) be concept names, C, D (or with C_i, D_i) (general) concepts, R (or with R_i) an abstract role, T (or with T_i) a concrete role, S (or with S_i) a concrete datatype d and lowercases (or with d_i) individual names, unless otherwise stated.

A *terminology* or a *TBox* $\mathcal{T}$ is a finite set of *general concept inclusion axioms* (GCIs) $C \sqsubseteq D$ (possibly contains nominals and datatypes in the language of $\mathcal{O}$). In an ABox, one describes a specific state of affairs of an application domain in terms of concept and roles. It is the statement about how concepts are related to each other. We use $C \equiv D$ as an abbreviation for the symmetrical pair of GCIs $C \sqsubseteq D$ and $D \sqsubseteq C$, called *concept definition*. An ABox $\mathcal{A}$ is a finite set of assertions of the forms $C(a)$ (*concept assertion*), $R(a, b)$ (role assertion), $a \doteq b$ (*equality assertion*), and $a \not\doteq b$ (*inequality assertion*). In general, axioms are GCIs, role axioms, concept assertions, role assertions, transitive axioms, equality assertions and inequality assertions. In an ABox, one describes a specific state of affairs of an application domain in terms of concepts and roles. A knowledge base (KB) $\mathcal{K}$ is a triple $(\mathcal{R}, \mathcal{T}, \mathcal{A})$.

The semantics is given by means of interpretations. A(n) (classical) interpretation $\mathcal{I}_c = (\Delta^{\mathcal{I}_c}, \cdot^{\mathcal{I}_c})$ consists of a non-empty domain $\Delta^{\mathcal{I}_c}$, disjoint from the concrete domain $\Delta_{\mathbf{D}}$, and a mapping $\cdot^{\mathcal{I}_c}$ which maps atomic and concepts, roles, and nominals according to Table 1 ($\sharp$ denotes set cardinality).

Note that there are an important more expressive concepts called *qualified number restrictions* ($\mathcal{Q}$), $\leqslant n\,R.C$ (*at most number restriction*) and $\geqslant n\,R.C$ (*at least number restrictions*) than number restrictions. Though qualified number restriction is not a constructor in $\mathcal{SHOIN}(\mathbf{D})$, we can apply $\mathcal{Q}$ to obtain more expressive DLs such as $\mathcal{SHOIQ}(\mathbf{D})$ and $\mathcal{SROIQ}(\mathbf{D})$ (i.e., the logical foundation of OWL 2 [26]). They are interpreted as follows:

$$(\leqslant n\, R.C)^{\mathcal{I}_c} = \left\{x \mid \sharp\left(\left\{y.(x, y) \in R^{\mathcal{I}_c} \wedge y \in C^{\mathcal{I}_c}\right\}\right) \leqslant n\right\}; \tag{2}$$

$$(\geqslant n\, R.C)^{\mathcal{I}_c} = \left\{x \mid \sharp\left(\left\{y.(x, y) \in R^{\mathcal{I}_c} \wedge y \in C^{\mathcal{I}_c}\right\}\right) \geqslant n\right\}. \tag{3}$$

An interpretation $\mathcal{I}_c$ *satisfies* a role inclusion axiom $R_1 \sqsubseteq R_2$ if and only if $R_1^{\mathcal{I}_c} \subseteq R_2^{\mathcal{I}_c}$, and it *satisfies* a transitivity axiom Trans(R) if and only if $R^{\mathcal{I}_c} = (R^{\mathcal{I}_c})^+$. An interpretation $\mathcal{I}_c$ *satisfies* an RBox $\mathcal{R}$ if and only if it satisfies each axiom in $\mathcal{R}$. In this case, $\mathcal{I}_c$ is named a *model* of $\mathcal{R}$, denoted by $\mathcal{I}_c \models \mathcal{R}$. An interpretation *satisfies* a GCI $C \sqsubseteq D$ if and only if $C^{\mathcal{I}_c} \subseteq D^{\mathcal{I}_c}$. An interpretation $\mathcal{I}_c$ *satisfies* a terminology $\mathcal{T}$ if it satisfies each axiom in $\mathcal{T}$. In this case, $\mathcal{I}_c$ is named a *model* of $\mathcal{T}$, denoted by $\mathcal{I}_c \models \mathcal{T}$. An interpretation $\mathcal{I}_c$ *satisfies* an *concept assertion* $C(a)$ (resp. a *role assertion* $R(a, b)$) if $a^{\mathcal{I}_c} \in C^{\mathcal{I}_c}$ (resp. $\langle a^{\mathcal{I}_c}, b^{\mathcal{I}_c}\rangle \in R^{\mathcal{I}_c}$). An assertion called *individual inequality* $a \not\approx b$ if $a^{\mathcal{I}_c} \neq b^{\mathcal{I}_c}$ for each interpretation $\mathcal{I}_c$. The *unique name assumption* (UNA), i.e., different names always referring to different entities, is not chosen but replaced by the individual inequality $\not\approx$. Thus, two nominals might refer to the same individuals.

An interpretation $\mathcal{I}_c$ *satisfies* an ABox $\mathcal{A}$ if it satisfies each assertion or individual inequalities in $\mathcal{A}$. In this case, $\mathcal{I}_c$ is named a *model* of $\mathcal{A}$, denoted by $\mathcal{I}_c \models \mathcal{A}$.

A concept C is *satisfiable* w.r.t. a role hierarchy $\mathcal{R}$ if there is a model $\mathcal{I}_c$ of $\mathcal{R}$ such that $C^{\mathcal{I}_c} \neq \emptyset$. A concept C is *satisfiable* w.r.t. a terminology $\mathcal{T}$ and a role hierarchy $\mathcal{R}$ if there is a model $\mathcal{I}_c$ of $\mathcal{T}$ and $\mathcal{R}$ such that $C^{\mathcal{I}_c} \neq \emptyset$. A KB is *coherent* if all of its concept names are satisfiable; and *incoherent* otherwise. A concept C is *subsumed* by a concept D w.r.t. a role hierarchy $\mathcal{R}$ if $C^{\mathcal{I}_c} \subseteq D^{\mathcal{I}_c}$ for each model $\mathcal{I}_c$ of $\mathcal{I}_c$.

An interpretation $\mathcal{I}_c$ is called *model* of a KB $\mathcal{K}$ if all $\mathcal{A}$, $\mathcal{T}$ and $\mathcal{R}$ are satisfied by $\mathcal{I}_c$. *Mod*$(\mathcal{K})$ is a collection of models of a KB $\mathcal{K}$. A KB $\mathcal{K}$ is *consistent* if there exists a model of $\mathcal{K}$. An ABox $\mathcal{A}$ is (*classically*) *consistent* w.r.t. a terminology $\mathcal{T}$ and a role hierarchy $\mathcal{R}$ if there is a model $\mathcal{I}_c$ of $\mathcal{T}$ and $\mathcal{R}$ which satisfies $\mathcal{A}$. A KB $\mathcal{K}$ entails a KB $\mathcal{K}'$ if $Mod(\mathcal{K}) \subseteq Mod(\mathcal{K}')$, denoted by $\mathcal{K} \models \mathcal{K}'$.

In DLs, there are two kinds of reasoning tasks, namely, *consistency problem* (whether a KB is consistent) and *entailment problem* (whether a KB entails an axiom) and entailment problem contains two subproblems: *instance checking* (checking whether a KB entails a concept assertion) and *subsumption* (checking whether a KB entails a GCI).

Indeed, entailment problems can be reduced into inconsistency checking problem.

Lemma 1. *(See [16].) Let $\mathcal{T}$ be a terminology, $\mathcal{R}$ a role hierarchy, $\mathcal{A}$ an ABox and C, D concepts. Let U be a transitive super-role of all roles occurring in $\mathcal{T}$ and their respective inverses but not occurring in $\mathcal{T}, C, D, \mathcal{A}$, or $\mathcal{R}$ (called* universal role*) where $U^{\mathcal{I}_c} = \Delta^{\mathcal{I}_c} \times \Delta^{\mathcal{I}_c}$ for any interpretation $\mathcal{I}_c$. We set*

$$\mathcal{R}_U := \mathcal{R} \cup \{R \sqsubseteq U \mid R \textit{ occurs in } \mathcal{T}, C, D, \mathcal{A}, \textit{ or } \mathcal{R}\}. \tag{4}$$

Then

(1) *$(\mathcal{T}, \mathcal{R}, \mathcal{A}) \models C(a)$ if and only if $(\mathcal{T}, \mathcal{R}, \mathcal{A} \cup \{\neg C(a)\})$ is inconsistent w.r.t. $\mathcal{R}_U$;*
(2) *$(\mathcal{T}, \mathcal{R}, \emptyset) \models C \sqsubseteq D$ if and only if $(\mathcal{T}, \mathcal{R}, \{(C \sqcap \neg D)(\iota)\})$ is inconsistent w.r.t. $\mathcal{R}_U$ for some new individual $\iota \in \Delta$.*

Similar to inference rules of the proof system in propositional logic, there exist some corresponding properties in DLs [24] as follows: let C, D, E be DL concepts,

(1) (**modus ponens**, MP) $\{C(a), C \sqsubseteq D\} \models D(a)$;
(2) (**modus tollens**, MT) $\{\neg D(a), C \sqsubseteq D\} \models C(a)$;
(3) (**disjunctive syllogism**, DS) $\{\neg C(a), (C \sqcup D)(a)\} \models D(a)$;
(4) (**resolution**) $\{(C \sqcup D)(a), (\neg C \sqcup E)(a)\} \models (D \sqcup E)(a)$;
(5) (**disjunction introduction**, DI) $\{C(a)\} \models (C \sqcup D)(a)$;
(6) (**intuitive equivalence**, IE) $\mathcal{K} \models C \sqsubseteq D$ if and only if $\mathcal{K} \models (\neg C \sqcup D)(a)$ for any $a \in N_I$;
(7) (**transitivity**) if $\mathcal{K}_1 \models \mathcal{K}_2$ and $\mathcal{K}_2 \models \mathcal{K}_3$ then $\mathcal{K}_1 \models \mathcal{K}_3$;
(8) (**excluded middle**, EM) $\emptyset \models (C \sqcup \neg C)(a)$.

DS is a special case of the property of resolution. IE can be used to reduce the subsumption problem can be reduced to the satisfaction problem

We say an entailment relation $\models_x$ satisfies a property **P** above if it is true when $\models$ is replaced by $\models_x$.

In the end of this section, we introduce two kinds of special axioms, namely, *tautology* and *contradiction* although they provide no valuable information for users. An axiom is a tautology if for each interpretation of an arbitrary non-empty domain, it satisfies that axiom (e.g., $\bot \sqsubseteq \top$, $\top(a)$, $(A \sqcup \neg A)(a)$, $(A \sqcap \neg A) \sqsubseteq (A \sqcup \neg A)$) and an axiom is a contradiction if there exists no any interpretation of some non-empty domain such that it satisfies that axiom (e.g., $\top \sqsubseteq \bot$, $\bot(a)$, $(A \sqcap \neg A)(a)$, $(A \sqcup \neg A) \sqsubseteq (A \sqcap \neg A)$). EM can capture some tautologies.

2.2. Paraconsistent logic and quasi-classical logic

2.2.1. Paraconsistent logic

In practical reasoning, it is common that there exists "too much" information (classically inconsistent information) about some situation. However, the reasoning of classical logic would be trivialized when treating inconsistent information because of a curious feature, known as *the principle of explosion* or (*ex falso quodlibet*) can be expressed formally as: for any formulas φ, ψ, $\{\varphi, \neg\varphi\} \models \psi$.

This is the need to derive reasonable inferences without deriving the trivial inferences that follow the ex falso quodlibet. In other words, we need a logic, called *paraconsistent logic* (or *inconsistency-tolerant logic*) where the principle of explosion fails in its reasoning [4].

Description logic fails to be paraconsistent because an inconsistent KB $\mathcal{K}$ does not possess any model, i.e., $Mod(\mathcal{K}) = \emptyset$. In this sense, we say that the entailment $\models$ satisfies the principle of explosion. Thus, the entailment of a paraconsistent description logic does not satisfy the principle of explosion, called a *paraconsistent entailment*.

Indeed, if some properties about inference are allowed together then conclusions inferred from inconsistent knowledge become explosive.

For instance, let $\models_p$ be an entailment. Assume that $\models_p$ satisfies DS, DI and transitivity. Given an ABox $\mathcal{A}_1 = \{A(a), \neg A(a)\}$, $\mathcal{A}_1$ is inconsistent. Because $\models_p$ satisfies DI, $\mathcal{A}_1 \models_p A \sqcup B(a)$ for arbitrary B. Let $\mathcal{A}_2 = \{A(a), \neg A(a), (A \sqcup B)(a)\}$. We conclude that $\mathcal{A}_1 \models_p \mathcal{A}_2$. Because $\models_p$ satisfies DS, we conclude that $\mathcal{A}_2 \models_p B(a)$. Then, $\{A(a), \neg A(a)\} \models_p B(a)$ for any B since $\models_p$ satisfies transitivity. In this sense, $\models_p$ loses a property so-called "*relevance*", which requires sharing of variables between premises and conclusion, in relevance logic [34].

A feasible method to make the principle of explosion invalid is weakening inference power by prohibiting some inference rules in reasoning [18].

2.2.2. Quasi-classical logic (QC logic)

QC logic, roughly taken a variant of Belnap's four-valued logic which is an important paraconsistent logic, presented in [19] exhibits the nice feature that no attention need to be paid to a special form that premises should have. An important feature that QC logic has stronger inference power than Belnap's four-valued logic [2].

Let p be an atom. p and $\neg p$ are *literals*. A *clause* is a set of literals. Let Δ be a set of literals (as a domain). We denote

$$\Delta^{\pm} = \{+l \mid l \in \Delta\} \cup \{-l \mid l \in \Delta\}; \tag{5}$$

where $+l$ is a positive object and $-l$ is a negative object.

For any non-empty $\mathcal{M} \subseteq \Delta^{\pm}$, $\mathcal{M}$ is called a *model* (or *possible world*) in Δ.

The meaning for positive and negative objects being in or out of some model are as follows: given a model $\mathcal{M}$ and a literal α,

- $+l \in \mathcal{M}$ means l is "satisfiable" in the model;
- $-l \in \mathcal{M}$ means $\neg l$ is "satisfiable" in the model;
- $+l \notin \mathcal{M}$ means l is not "satisfiable" in the model;
- $-l \notin \mathcal{M}$ means $\neg l$ is not "satisfiable" in the model.

Two satisfiability relations, namely, *strong satisfaction* ($\models_s$) and *weak satisfaction* ($\models_w$), are introduced as follows: let Δ be a domain, $l, l_1, \ldots, l_n$ literals, $Cl, Cl_1, \ldots, Cl_m$ clauses and $\mathcal{M}$ a model in Δ,

- $\models_s$ is defined as follows:
  (1) $\mathcal{M} \models_s l$ if $+l \in \mathcal{M}$;
  (2) $\mathcal{M} \models_s \neg l$ if $-l \in \mathcal{M}$;
  (3) $\mathcal{M} \models_s \{l_1, \ldots, l_n\}$ if $\mathcal{M} \models_s l_i$ for all $i \in \{1, \ldots, n\}$;
  (4) $\mathcal{M} \models_s \{Cl_1, \ldots, Cl_m\}$ if $\mathcal{M} \models_s Cl_i$ for all $i \in \{1, \ldots, m\}$;
  (5) $\mathcal{M} \models_s Cl_1 \vee \cdots \vee Cl_m$ if and only if (($\mathcal{M} \models_s Cl_1$ or ... or $\mathcal{M} \models_s Cl_m$) and (if $\mathcal{M} \models_s \neg l_i$ for some $i \in \{1, \ldots, n\}$ where $Cl_j = \{l_1, \ldots, l_n\}$ then $\mathcal{M} \models_s Cl_1 \vee \cdots \vee Cl_{i-1} \vee Cl_{i+1} \vee \cdots \vee Cl_m$)) (for some $j \in \{1, \ldots, m\}$).
- $\models_w$ is defined as follows:
  (1) $\mathcal{M} \models_w Cl$ if $\mathcal{M} \models_s Cl$ for any clause Cl;
  (2) $\mathcal{M} \models_w Cl_1 \vee Cl_2$ if $\mathcal{M} \models_s Cl_i$ ($i = 1$ or 2);
  (3) $\mathcal{M} \models_w \{Cl_1, \ldots, Cl_m\}$ if $\mathcal{M} \models_w Cl_i$ for all $i \in \{1, \ldots, m\}$.

Let Δ be a domain and $Cl, Cl_1, \ldots, Cl_m$ be clauses in Δ. The quasi-classical entailment (or QC entailment), denoted by $\models_Q$, is defined as follows: $\{Cl_1, \ldots, Cl_m\} \models_Q Cl$ if for all models $\mathcal{M}$ in Δ, $\mathcal{M} \models_s Cl_1, \ldots, \mathcal{M} \models_s Cl_m$ implies $\mathcal{M} \models_w Cl$.

Let K be a set of formulas (as a KB). Every formula $\varphi \in \mathcal{K}$ is equivalent to the disjunction form of clauses $C_1 \vee \cdots \vee C_m$ where C_i is a clause ($i \in \{1, \ldots, m\}$). Therefore, the QC entailment is indeed also introduced between two KBs.

The following example illustrates that QC logic is paraconsistent. That is, the rule of ex falso quodlibet fails in QC logic. Let $C = \{p, \neg p\}$ be a clause and $\Delta = \{p, q\}$. Let $\mathcal{M} = \{+p, -p\}$ be a model of Δ. However, $\mathcal{M} \models C$ while $\mathcal{M} \not\models_w q$ because $+q \notin \mathcal{M}$.

Different from the four-valued entailment, the QC entailment satisfies the modus ponens rule, that is, $\{\varphi \vee \psi, \neg\varphi\} \models \psi$ for any formulas φ, ψ. For instance, $\Delta = \{p, q\}$ and $\mathcal{K} = \{p \vee q, \neg p\}$ be a KB. Let $C_1 = \{p\}, C_2 = \{q\}$ and $C_3 = \{\neg p\}$ be clauses. For all models $\mathcal{M}$ in Δ, if $\mathcal{M} \models_s C_1 \vee C_2$ and $\mathcal{M} \models_s C_3$ where $C_3 = \{\neg p\}$ and $C_1 = \{p\}$ then $\mathcal{M} \models C_2$. Therefore, $\mathcal{K} \models q$.

3. Using QCDL to improve paraconsistent reasoning with DL

Based on four-valued DL, this section aims to introduce a paraconsistent version of DL, called *quasi-classical DL* (briefly, QCDL) and employs QCDL to tolerate inconsistencies occurring DL KBs.

3.1. Quasi-classical description logic

The QC logic presented by [19] is built on Belnap's four-valued logic [2]. In four-valued DL, the negation of a concept $\neg C$ is no longer taken as its "opposite" concept of C but a different concept from C. The "opposite" concept is important to represent many properties about inference.

To represent the "opposite" concept of a given concept, we need to introduce a weaker version of concept negation called the *quasi-classical negation* (QC negation) of a concept. The QC negation of a concept C is denoted $\overline{C}$.

Thus QCDL extends the syntax of classical DLs slightly.

The QC negation is inspired from a so-called *total negation* [23] which is also introduced in rough description logic [7] with different meanings.

Intuitively, the QC negation reverses both the information of being true and of being false.

Axioms are a special kind of QC axioms. Thus $\mathcal{A}$ and $\mathcal{T}$ are an ABox and a terminology of QCDLs respectively. In this case, we also say $\mathcal{A}$, $\mathcal{T}$, $\mathcal{R}$ and $\mathcal{K}$ a *QC ABox*, a *QC terminology* or a *QC TBox* and a *QC role hierarchy* or a *QC KB* respectively. In other word, each KB in DLs is also a QC KB without the QC negation.

For instance, let $\mathcal{A} = \{Penguin(tweety), \overline{\neg Bird}(tweety), \neg Fly(tweety), \exists HasChild.Penguin(tweety)\}$ be a QC ABox and $\mathcal{T} = \{Bird \sqsubseteq \overline{\neg Fly}\}$ be a QC TBox. Two new axioms $\overline{\neg Bird}(tweety)$ (*tweety* is known not to be a non-bird) and $Bird \sqsubseteq \overline{\neg Fly}$ (all birds are known not to be flightless) are called a *QC concept assertion* and a *QC GCI*. However, $\overline{\neg Bird}(tweety)$ does not mean that *tweety* is necessarily known to be a member of *Bird* under our proposal non-classical semantics.

In syntax, concept descriptions in QC-$\mathcal{SHOIN}(\mathbf{D})$ are formed according to the following syntax rule:

$$C, D \to A \mid d \mid \top \mid \bot \mid \neg C \mid C \sqcup D \mid C \sqcap D \mid \exists R.C \mid \forall R.C \mid \overline{C} \mid \{o\} \mid \geqslant nR \mid \leqslant nR \mid \exists T.d \mid \forall T.d. \tag{6}$$

A novelty in our approach is that we propose to use two types of interpretations for defining our proposal QC semantics: *weak interpretations* and *strong interpretations*. Different from standard interpretations in DLs where each concept (or role) is mapped to a set of instances, the two types of interpretations will map every concept to a *pair* of sets of instances, where the former characterizes those instances certain to belong to the concept, and the latter characterizes those instances certain not to belong to the concept. Weak interpretations are essentially an extension of the well-known four-valued interpretations.

Before we introduce these two types of interpretations, we first define *base interpretations*.

Definition 1. A *base interpretation* $\mathcal{I}$ is a pair $(\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ where the domain $\Delta^{\mathcal{I}}$ is a set of individuals, $\Delta_{\mathbf{D}}$ a concrete domain of datatypes and the assignment function $\cdot^{\mathcal{I}}$ assigns each individuals to an element of $\Delta^{\mathcal{I}}$ and assigns

$$\begin{aligned}
\top^{\mathcal{I}} &= \langle \Delta^{\mathcal{I}}, \emptyset \rangle; \\
\bot^{\mathcal{I}} &= \langle \emptyset, \Delta^{\mathcal{I}} \rangle; \\
A^{\mathcal{I}} &= \langle +A, -A \rangle; \\
R^{\mathcal{I}} &= \langle +R, -R \rangle; \\
\left(R^{-}\right)^{\mathcal{I}} &= \langle +R^{-}, -R^{-} \rangle; \\
\left(R^{tc}\right)^{\mathcal{I}} &= \langle +R^{tc}, -R^{tc} \rangle; \\
d^{\mathcal{I}} &= \langle +d, -d \rangle; \\
T^{\mathcal{I}} &= \langle +T, -T \rangle;
\end{aligned} \tag{7}$$

where $\pm A, N \subseteq \Delta^{\mathcal{I}}$, $\pm R, \pm R^{-}, \pm R^{tc} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$, $\pm T \subseteq \Delta^{\mathcal{I}} \times \Delta_{\mathbf{D}}$ and $\pm d^{\mathbf{D}} \subseteq \Delta_{\mathbf{D}}$.

Note that $+X$ and $-X$ are not necessarily disjoint in Definition 1 when $X \in \{C, R, T, d\}$. Intuitively, $+X$ is the set of elements known to be in the extension of X while $-X$ is the set of elements known to be not in the extent of X.

For instance, let $\mathcal{I}$ be a base interpretation, assume that $\cdot^{\mathcal{I}}$ assigns *Student* to a pair $\langle\{Jack\}, \{Wade\}\rangle$. The interpretation $\mathcal{I}$ tells that *Jack* is known to be a student and *Wade* is known not to be a student (*Wade* is possibly a staff).

Though the QC negation is different from the negation under base interpretations, they are identical under (classical) two-valued interpretations.

A datatype possibly contains inconsistent information.

For instance, given three assertions *Integer*(2012), *String*(2012) and $Integer \sqcap String \sqsubseteq \bot$, the first assertion states that 2012 is in the value space of integer (e.g., the 2012th year), the second assertion states 2012 is in the value space of string (e.g., the film titled "2012") and the third axiom states that the value space of integer and the value space of string are disjoint. As a result, they cause inconsistency.

A base interpretation of datatype *Integer* and *String*, which are pairs $\langle +Integer^{\mathbf{D}}, -Integer^{\mathbf{D}}\rangle$ and $\langle +String^{\mathbf{D}}, -String^{\mathbf{D}}\rangle$ where the instance $2012 \in +Integer^{\mathbf{D}} \cap -Integer^{\mathbf{D}}$ and $2012 \in +String^{\mathbf{D}} \cap -String^{\mathbf{D}}$ respectively, could tolerate such three assertions in a KB. Thus our base interpretation of a datatype is necessary and reasonable.

The QC semantics is characterized by two new interpretations, namely, *weak interpretations* and *strong interpretations*, which are built on base interpretations defined in Definition 1.

Definition 2. A *weak interpretation* $\mathcal{I}$ is a base interpretation $(\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ such that the assignment function $\cdot^{\mathcal{I}}$ satisfies the conditions as follows:

$$\begin{aligned}
(C \sqcap D)^{\mathcal{I}} &= \langle +C \cap +D, -C \cup -D\rangle;\\
(C \sqcup D)^{\mathcal{I}} &= \langle +C \cup +D, -C \cap -D\rangle;\\
(\neg C)^{\mathcal{I}} &= \langle -C, +C\rangle;\\
(\bar{C})^{\mathcal{I}} &= \langle \Delta^{\mathcal{I}} \setminus +C, \Delta^{\mathcal{I}} \setminus -C\rangle;\\
(\exists R.C)^{\mathcal{I}} &= \langle \{x \mid \exists y, \langle x, y\rangle \in +R \text{ and } y \in +C\}, \{x \mid \forall y.\langle x, y\rangle \in +R \text{ implies } y \in -C\}\rangle;\\
(\forall R.C)^{\mathcal{I}} &= \langle \{x \mid \forall y, \langle x, y\rangle \in +R \text{ implies } y \in +C\}, \{x \mid \exists y.\langle x, y\rangle \in +R \text{ and } y \in -C\}\rangle;\\
\{o\}^{\mathcal{I}} &= \langle \{o^{\mathcal{I}}\}, N\rangle \quad \text{where } N \subseteq \Delta^{\mathcal{I}};\\
(\neg d)^{\mathcal{I}} &= \langle -d^{\mathbf{D}}, +d^{\mathbf{D}}\rangle \quad \text{where } d^{\mathcal{I}} = \langle +d^{\mathbf{D}}, -d^{\mathbf{D}}\rangle;\\
(\bar{d})^{\mathcal{I}} &= \langle \Delta_{\mathbf{D}} \setminus +d^{\mathbf{D}}, \Delta_{\mathbf{D}} \setminus -d^{\mathbf{D}}\rangle \quad \text{where } d^{\mathcal{I}} = \langle +d^{\mathbf{D}}, -d^{\mathbf{D}}\rangle;\\
(\geqslant nR)^{\mathcal{I}} &= \langle \{x \mid \sharp(\{y.\langle x, y\rangle \in +R\}) \geqslant n\}, \{x \mid \sharp(\{y.\langle x, y\rangle \in +R\}) < n\}\rangle;\\
(\leqslant nR)^{\mathcal{I}} &= \langle \{x \mid \sharp(\{y.\langle x, y\rangle \in +R\}) \leqslant n\}, \{x \mid \sharp(\{y.\langle x, y\rangle \in +R\}) > n\}\rangle;\\
(\exists T.d)^{\mathcal{I}} &= \langle \{x \in \Delta^{\mathcal{I}} \mid \exists y.\langle x, y\rangle \in +T \text{ and } y \in +d^{\mathbf{D}}\}, \{x \in \Delta^{\mathcal{I}} \mid \forall y.\langle x, y\rangle \in +T \text{ implies } y \in -d^{\mathbf{D}}\}\rangle;\\
(\forall T.d)^{\mathcal{I}} &= \langle \{x \in \Delta^{\mathcal{I}} \mid \forall y.\langle x, y\rangle \in +T \text{ implies } y \in +d^{\mathbf{D}}\}, \{x \in \Delta^{\mathcal{I}} \mid \exists y.\langle x, y\rangle \in +T \text{ and } y \in -d^{\mathbf{D}}\}\rangle.
\end{aligned}$$

Indeed, weak interpretations extend four-valued interpretations in the QC negation of concepts.

Intuitively speaking, based on weak interpretations, an individual a is known to be an instance of $+(C \sqcup D)$ if and only if

(1) a is an instance of $+C$ or a is an instance of $+D$; and,
(2) a is an instance of $-C$ and a is an instance of $-D$.

Intuitively speaking, $Student \sqcup Staff$ represents all members who are students or staffs. Assume that *Jack* is an instance of *Student* but *Jack* is an instance of *Staff*; and *Wade* is an instance of *Staff* but *Wade* is an instance of $\neg Student$. Thus $Jack, Wade \in +(Fly \sqcup Bird)$ and $Jack, Wade \notin -(Student \sqcup Staff)$.

Moreover, a is known to be an instance of $+(C \sqcap D)$ if and only if

(1) a is an instance of $+C$ and a is an instance of $+D$; and,
(2) a is an instance of $-C$ or a is an instance of $-D$.

For instance, $Fly \sqcap Bird$ represents all birds who can fly. Assume that *swallow* is an instance of *Fly* and *swallow* is an instance of *Bird*. Thus $swallow \in +(Fly \sqcap Bird)$. Assume that *tweety* is an instance of $\neg Fly$ or *tweety* is an instance of $\neg Bird$. Thus $tweety \in -(Fly \sqcap Bird)$.

Furthermore, inconsistent phenomenon also appears in nominals mainly because there exist different cognitions within a same subject among persons. For instance, $\{a(b), a \not\doteq b\}$. From definition of weak interpretation on a nominal $\{o\}$, different subset N of the domain is corresponding to different base interpretation. If $N \cap \{o\} \neq \emptyset$ then there is an inconsistency in those nominals. Those nominals with multiple individuals $\{o_1, \dots, o_m\}$ are still used to denote to the disjunction of nominals $\{o_1\} \sqcup \dots \sqcup \{o_m\}$ in our QCDL following from [24].

Note that the weak interpretations in qualified number restrictions (for short, $\mathcal{Q}$) in forms of $\leqslant n\,R.C$ and $\geqslant n\,R.C$ are also defined as follows:

$$(\leqslant nR.C)^{\mathcal{I}} = \big\langle \big\{x \mid \sharp\big(\big\{y.\langle x, y\rangle \in +R \wedge y \notin -C\big\}\big) \leqslant n\big\}, \big\{x \mid \sharp\big(\big\{y.\langle x, y\rangle \in +R \wedge y \in +C\big\}\big) \geqslant n+1\big\}\big\rangle; \tag{8}$$

$$(\geqslant nR.C)^{\mathcal{I}} = \big\langle \big\{x \mid \sharp\big(\big\{y.\langle x, y\rangle \in +R \wedge y \in +C\big\}\big) \geqslant n\big\}, \big\{x \mid \sharp\big(\big\{y.\langle x, y\rangle \in +R \wedge y \notin -C\big\}\big) \leqslant n-1\big\}\big\rangle; \tag{9}$$

where $R^{\mathcal{I}} = \langle +R, -R\rangle$ and $C^{\mathcal{I}} = \langle +C, -C\rangle$.

By using qualified number restrictions, we can technically extend our scenario in more expressive DLs such as $\mathcal{SHOIQ}(\mathbf{D})$. In this paper, we still use $\mathcal{SHOIN}(\mathbf{D})$ to simplify our discussion and make readers understand our technique clear.

Without confusion, we also denote $\overline{+C} = \Delta^{\mathcal{I}} \setminus +C$ and $\overline{-C} = \Delta^{\mathcal{I}} \setminus -C$, i.e., $(\overline{C})^{\mathcal{I}} = \langle \overline{+C}, \overline{-C}\rangle$, where $C^{\mathcal{I}} = \langle +C, -C\rangle$.

A satisfaction relation determined by weak interpretations, denoted by $\models_w$, is defined as follows.

Definition 3. Let $\mathcal{I}$ be a weak interpretation. A *weak satisfaction* ($\models_w$) is defined as follows: let $X^{\mathcal{I}} = \langle +X, -X\rangle$ where $X \in \{C, D, d, R, S\}$,

(1) $\mathcal{I} \models_w C(a)$ if $a^{\mathcal{I}} \in +C$;
(2) $\mathcal{I} \models_w C \sqsubseteq D$, if $+C \subseteq +D$;
(3) $\mathcal{I} \models_w R(a, b)$, if $\langle a^{\mathcal{I}}, b^{\mathcal{I}}\rangle \in +R$;
(4) $\mathcal{I} \models_w d_1 \sqsubseteq d_2$ if $+d_1 \subseteq +d_2$;
(5) $\mathcal{I} \models_w R \sqsubseteq S$ if $+R \subseteq +S$, for any role $R, S \in \mathbf{R}_A$ or $R, S \in \mathbf{R}_D$;
(6) $\mathcal{I} \models_w \mathrm{Trans}(R)$ if $+R = (+R)^{tc}$, for any abstract $R, S \in \mathbf{R}_A$;
(7) $\mathcal{I} \models_w a \doteq b$ if $a^{\mathcal{I}} = b^{\mathcal{I}}$;
(8) $\mathcal{I} \models_w a \not\doteq b$ if $a^{\mathcal{I}} \neq b^{\mathcal{I}}$.

Besides the weak satisfaction $\models_w$ extends the four-valued satisfaction $\models_4$ in QC axioms, $\models_w$ is also different from $\models_4$ in GCIs. There exist three kinds of four-valued satisfactions on GCIs, namely, *material GCI* ($C \mapsto D$), *internal GCI* ($C \sqsubset D$) and *strong GCI* ($C \to D$), which are formally defined as follows: let $\mathcal{I}_4$ be a four-valued interpretation and C, D be two concepts without QC negation,

(1) $\mathcal{I}_4 \models_4 C \mapsto D$, if $\overline{-C} \subseteq +D$;
(2) $\mathcal{I}_4 \models_4 C \sqsubset D$, if $+C \subseteq +D$;
(3) $\mathcal{I}_4 \models_4 C \to D$, if $-D \subseteq -C$.

Intuitively, (1) material GCI is cautious in the sense that contradictory information is not propagated. For instance, *Healthy* $\mapsto$ *MarathonParticipant* which is supposed to say that somebody (i.e. a person who has signed up for a run) participates in a marathon if he checks out to be healthy; (2) internal inclusion propagates contradictory information forward, but not backward as it does not allow for contraposition reasoning. It can thus be characterized as a brave way of handling inconsistency. It should be used whenever it is important to infer the consequent even if the antecedent may be contradictory. In a paraconsistent context, the axiom is thus best modeled by means of internal inclusion, i.e. as *OilLeakage* $\sqsubset$ *RobotMalfunction*; (3) strong GCI respects the deduction theorem and contraposition reasoning. In a paraconsistent context, it is thus the inclusion to be used for universal truth, such as *Square* $\to$ *FourEdged* (see [23]).

The weak satisfaction on GCIs adopts the four-valued internal semantics for GCIs, since the motivation of QC logic is enhancing the inference power. Moreover, the QC negation exactly satisfies the property of intuitive equivalence under internal GCIs (discussed later).

Note that for any weak interpretation $\mathcal{I}$, there exists no longer some connection between $\mathcal{I} \models_s \overline{C}(a)$ and $\mathcal{I} \models_s \neg C(a)$. The former states that a is known not to be an instance of C while the latter states that a is known to be an instance of $\neg C$.

For instance, let $\mathcal{I}$ be a weak interpretation and $\Delta^{\mathcal{I}} = \{tweety, fred, bee\}$ such that $Fly^{\mathcal{I}} = \langle\{bee, tweety\}, \{tweety\}\rangle$. Thus $\mathcal{I} \not\models_w \overline{Fly}(bee)$ and $\mathcal{I} \not\models_w \neg Fly(bee)$ while $\mathcal{I} \not\models_w \neg Fly(tweety)$ and $\mathcal{I} \models_w \neg Fly(tweety)$.

Let $\mathcal{K}$ be a QC KB. A base interpretation $\mathcal{I}$ is called a *weak model* of $\mathcal{K}$ if for all axiom ϕ in $\mathcal{K}$, $\mathcal{I} \models_w \phi$. $Mod^w(\mathcal{K})$ denotes the collection of all weak models of $\mathcal{K}$.

Given a QC KB $\mathcal{K}$ and a QC axiom ϕ, $\mathcal{K}$ *w-entails* ϕ, denoted $\mathcal{K} \models_w \phi$, if for each base interpretation $\mathcal{I}$, $\mathcal{I} \models_w \mathcal{K}$ implies $\mathcal{I} \models_w \phi$, i.e., $Mod^w(\mathcal{K}) \subseteq Mod^w(\{\phi\})$.

Besides, $\models_w$ satisfies DI and transitivity while it does not satisfy EM and IE.

Proposition 1. *Let C, D be two DL concepts.* $\{C(a)\} \models_w (C \sqcup D)(a)$.

Proof. For any base interpretation $\mathcal{I}$, $\mathcal{I} \models_w C(a)$ if and only if $a^{\mathcal{I}} \in +C$ where $C^{\mathcal{I}} = \langle +C, -C\rangle$. Thus $a^{\mathcal{I}} \in (+C \cup +D)$ where $D^{\mathcal{I}} = \langle +D, -D\rangle$. Then $\mathcal{I} \models_w (C \sqcup D)(a)$. Therefore, $\{C(a)\} \models_w (C \sqcup D)(a)$. $\square$

Though $\models_w$ satisfies MP, it does not satisfies MT and DS. To recover these logical properties, we first introduce *strong interpretation* and then define our paraconsistent entailment relation in terms of both weak and strong satisfaction relations.

Definition 4. A *strong interpretation* $\mathcal{I}$ is a base interpretation $(\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ such that the assignment function $\cdot^{\mathcal{I}}$ satisfies the conditions in Definition 2 except that the conjunction and the disjunction of concepts are interpreted as follows: let $C^{\mathcal{I}} = \langle +C, -C\rangle$ and $D^{\mathcal{I}} = \langle +D, -D\rangle$,

$$(C \sqcap D)^{\mathcal{I}} = \big\langle (+C \cap +D), (-C \cup -D) \cap (-C \cup \overline{+D}) \cap (\overline{+C} \cup -D)\big\rangle; \tag{10}$$

$$(C \sqcup D)^{\mathcal{I}} = \big\langle (+C \cup +D) \cap (\overline{-C} \cup +D) \cap (+C \cup \overline{-D}), (-C \cap -D)\big\rangle. \tag{11}$$

Compared with the weak interpretation, the strong interpretation of disjunction of concepts tightens the condition that an individual is known to belong to a concept.

Intuitively speaking, based on strong interpretations, an individual a is known to be an instance of $+(C \sqcup D)$ if and only if

(1) a is an instance of $+C$ or a is an instance of $+D$;
(2) if a is also an instance of $-C$ then a must be an instance of $+D$;
(3) if a is also an instance of $-D$ then a must be an instance of $+C$.

The strong interpretation of conjunction of concepts is defined by relaxing the condition that an individual is known to be not contained in the extension of a concept.

Intuitively speaking, based on strong interpretations, an individual a is known to be not contained in the extension of $+(C \sqcap D)$ if and only if

(1) a is an instance of $-C$ or a is an instance of $-D$;
(2) if a is known to be an instance of $+C$ then a must be an instance of $-D$;
(3) if a is an instance of $+D$ then a must be an instance of $-C$.

For instance, let $\mathcal{I}_i$ be two base interpretations ($i = 1, 2$) such that $\Delta^{\mathcal{I}} = \{Jack, Wade, Mary\}$. Assume that $Student^{\mathcal{I}_i} = \langle\{Jack\}, \{Wade\}\rangle$ and $(Student \sqcup Staff)^{\mathcal{I}_i} = \langle\{Jack, Wade\}, \emptyset\rangle$ ($i = 1, 2$). Moreover, $Staff^{\mathcal{I}_1} = \langle\emptyset, \{Jack\}\rangle$ and $Staff^{\mathcal{I}_2} = \langle\{Wade\}, \{Jack\}\rangle$. Let's consider the two interpretations.

(1) For interpretation $\mathcal{I}_1$, $(Student \sqcup Staff)^{\mathcal{I}_1} = \langle\{Jack, Wade\}, \emptyset\rangle$ while $\langle(+Student \cup +Staff) \cap (\overline{-Student} \cup +Staff) \cap (+Student \cup \overline{-Staff}), -Student \cap -Staff\rangle = \langle\{Jack\}, \emptyset\rangle$ since $+Student = \{Jack\}$, $-Student = \{Wade\}$, $+Staff = \emptyset$ and $-Staff = \{Jack\}$. Thus Eq. 11 in Definition 4 does not hold.
(2) For interpretation $\mathcal{I}_2$, $(Student \sqcup Staff)^{\mathcal{I}_1} = \langle\{Jack, Wade\}, \emptyset\rangle$ while $\langle(+Student \cup +Staff) \cap (\overline{-Student} \cup +Staff) \cap (+Student \cup \overline{-Staff}), -Student \cap -Staff\rangle = \langle\{Jack, Wade\}, \emptyset\rangle$ since $+Student = \{Jack\}$, $-Student = \{Wade\}$, $+Staff = \{Wade\}$ and $-Staff = \{Jack\}$. Thus Eq. (10) in Definition 4 holds.

Then $\mathcal{I}_2$ is a strong interpretation of $Student \sqcup Staff$ while $\mathcal{I}_1$ is not a strong interpretation but a weak interpretation of $Student \sqcup Staff$. $\mathcal{I}_2$ states that *Wade* is a staff while $\mathcal{I}_1$ states that *Wade* is not a staff. Intuitively, $\mathcal{I}_2$ provides much information than $\mathcal{I}_1$ about whether *Wade* is a staff.

Similarly, we can define the strong satisfaction relation, denoted by $\models_s$, in terms of strong interpretations.

Definition 5. Let $\mathcal{I}$ be a strong interpretation. A *strong satisfaction* ($\models_s$) is defined as the same of the weak satisfaction on axioms except for concept/role inclusions as follows: let $X^{\mathcal{I}} = \langle +X, -X\rangle$ where $X \in \{C, D, d, d_1, d_2\}$,

(1) $\mathcal{I} \models_s C \sqsubseteq D$, if $\overline{-C} \subseteq +D$, $+C \subseteq +D$ and $-D \subseteq -C$;
(2) $\mathcal{I} \models_s d_1 \sqsubseteq d_2$ if $\overline{-d_1} \subseteq +d_2$, $+d_1 \subseteq +d_2$ and $-d_2 \subseteq -d_1$;
(3) $\mathcal{I} \models_s R \sqsubseteq S$ if $\overline{-R} \subseteq +S$, $+R \subseteq +S$ and $-S \subseteq -R$, for any role $R, S \in \mathbf{R}_A$ or $R, S \in \mathbf{R}_D$.

From Definition 3 and Definition 5, the definition of $\models_s$ hardly differs from the definition of $\models_w$ except GCIs where both nominals inclusions and datatype inclusions are technically taken as GCIs.

Role axioms $R \sqsubseteq S$ are also interpreted under the strong satisfaction analogous to GCIs. In general, the negation of role in form of $\neg R$ is not yet a syntactical constructor in $\mathcal{SHOIN}(\mathbf{D})$ [1]. In this sense, there exists no inconsistency caused by both $R(a, b)$ and $\neg R(a, b)$. However, there exists still some inconsistency caused by roles. For instance, $(\geqslant 3\, hasChild \sqcap \leqslant 2\, hasChild)(Tom)$ is a contradiction.

Given an object X ($X \in \{C, d, R, T\}$) and a strong interpretation $\mathcal{I}$, X is called *clash-free* w.r.t. $\mathcal{I}$ if $+X$ and $-X$ are complementary (i.e., $+X = \overline{-X}$ and $-X = \overline{+X}$) where $X^{\mathcal{I}} = \langle +X, -X\rangle$.

As a result, the weak satisfaction is equivalent to the strong satisfaction in the clash-free case.

Proposition 2. *Let X_i be an object ($X_i \in \{C_i, d_i, R_i, T_i\}$) ($i = 1, 2$). For any base interpretation $\mathcal{I}$, if X_1, X_2 are clash-free w.r.t. $\mathcal{I}$ then we have $\mathcal{I} \models_w X_1 \sqsubseteq X_2$ if and only if $\mathcal{I} \models_s X_1 \sqsubseteq X_2$.*

Proof.

$$\begin{aligned}
\mathcal{I} \models_s X_1 \sqsubseteq X_2 \quad &\Leftrightarrow \quad \overline{-X_1} \subseteq +X_2, \quad +X_1 \subseteq +X_2, \quad -X_2 \subseteq -X_1; \\
&\Leftrightarrow \quad +X_1 \subseteq +X_2, \quad -X_2 \subseteq -X_1 \quad \text{since } +X_1 = \overline{-X_1}; \\
&\Leftrightarrow \quad +X_1 \subseteq +X_2 \quad \text{since } -X_2 = \overline{+X_2}; \\
&\Leftrightarrow \quad \mathcal{I} \models_w X_1 \sqsubseteq X_2.
\end{aligned}$$

Here "$\Leftrightarrow$" denotes "if and only if" in this paper. □

The strong satisfaction on GCIs is defined by combining three kinds of four-valued satisfactions on GCIs. The satisfaction not only preserves the property of intuitive equivalence (see Proposition 4) but also avoids to diffuse inconsistent knowledge.

For instance, let $\mathcal{I}_i$ be two base interpretations ($i = 1, 2$) such that $\Delta^{\mathcal{I}} = \{Jack, Wade, Mary\}$. Assume that $PhDStudent^{\mathcal{I}_i} = \langle\{Jack, Mary\}, \{Mary, Wade\}\rangle$ ($i = 1, 2$), $Student^{\mathcal{I}_1} = \langle\{Jack, Mary\}, \{Mary, Wade\}\rangle$ and $Student^{\mathcal{I}_2} = \langle\{Jack, Mary\}, \{Wade\}\rangle$. Then $\mathcal{I}_1 \not\models_s PhDStudent \sqsubseteq Student$ while $\mathcal{I}_2 \models_s PhDStudent \sqsubseteq Student$. Thus $\mathcal{I}_1$ contains an inconsistency about whether *Mary* is a PhD student while $\mathcal{I}_2$ does not contain any inconsistency about *Mary* is a student. Intuitively, $\mathcal{I}_2$ prevents some inconsistency to be spread from some class to its superclass through $\sqsubseteq$. In a short, the strong satisfaction on GCIs is reasonable.

Analogously, for any interpretation $\mathcal{I}$, there exists no longer some connection between $\mathcal{I} \models_s \bar{C}(a)$ and $\mathcal{I} \models_s \neg C(a)$.

For instance, let $\mathcal{I}$ be a strong interpretation and $\Delta^{\mathcal{I}} = \{tweety, fred, bee\}$ such that $Fly^{\mathcal{I}} = \langle\{bee, tweety\}, \{tweety\}\rangle$. Thus $\mathcal{I} \not\models_s \overline{Fly}(tweety)$ and $\mathcal{I} \models_s \neg Fly(tweety)$ while $\mathcal{I} \not\models_s \neg Fly(fred)$ and $\mathcal{I} \models_s \overline{Fly}(fred)$.

Let $\mathcal{K}$ be a QC KB. A base interpretation $\mathcal{I}$ is called a *strong model* of $\mathcal{K}$ if for all axiom ϕ in $\mathcal{K}$, $\mathcal{I} \models_s \phi$. $Mod^s(\mathcal{K})$ denotes the collection of all strong models of $\mathcal{K}$.

Given a QC KB $\mathcal{K}$ and a QC axiom ϕ, $\mathcal{K}$ *s-entails* ϕ, denoted $\mathcal{K} \models_s \phi$, if for each base interpretation $\mathcal{I}$, $\mathcal{I} \models_s \mathcal{K}$ implies $\mathcal{I} \models_s \phi$, i.e., $Mod^s(\mathcal{K}) \subseteq Mod^s(\{\phi\})$.

Next, we discuss that the relationship between the weak satisfaction and the strong satisfaction is shown in the following proposition.

Proposition 3. *Let $\mathcal{I}$ be a base interpretation and ϕ be a QC axiom. If $\mathcal{I} \models_s \phi$ then $\mathcal{I} \models_w \phi$.*

Proof. If we technically treat nominals and datatypes as concepts, then a QC axiom ϕ can have five forms: $R(a, b)$, $Trans(R)$, $C \sqsubseteq D$, $R \sqsubseteq S$ and $C(a)$.

(1) When ϕ is $R(a, b)$, $Trans(R)$, $C \sqsubseteq D$ or $R \sqsubseteq S$ it is easy to show that the theorem holds by Definition 3.

(2) When ϕ is $C(a)$, there are three cases, namely, C is an atomic concept, a datatype and a complex concept. This theorem clearly holds when C is an atomic concept. In the following, we mainly discuss the case that C is a complex concept by induction over the number n of connectives and quantifiers in C.

  (a) (Base step) When $n = 1$, C can be the following ten forms, namely, $\neg A$, $\overline{A}$, d, $\bar{d}$, $D \sqcap E$, $D \sqcup E$, $\forall R.D$, $\exists R.D$, $\forall T.d$ and $\exists T.d$ where A is an atomic concept, D, E are concepts, d is a datatype, T is a concrete role, R is an abstract role and a, b are individuals.

    (i) When ϕ has one of the following forms, namely, $\neg A(a)$, $\overline{A}(a)$, $d(a)$, $\bar{d}(a)$, $\forall R.D(a)$, $\exists R.D(a)$, $\forall T.d$ and $\exists T.d$, the strong interpretation of ϕ is equivalent to the weak interpretation of ϕ by Definition 1 and Definition 2. Therefore, this theorem clearly holds.

    (ii) Suppose $\phi = (C \sqcap D)(a)$. If $\mathcal{I} \models_s (C \sqcap D)(a)$ then $a^{\mathcal{I}} \in (+C \cap +D)$ by Definition 2 where $C^{\mathcal{I}} = \langle +C, -C\rangle$ and $D^{\mathcal{I}} = \langle +D, -D\rangle$. Therefore, $\mathcal{I} \models_w (C \sqcap D)(a)$ by Definition 2.

    (iii) Suppose $\phi = (C \sqcup D)(a)$. If $\mathcal{I} \models_s (C \sqcup D)(a)$ then $a^{\mathcal{I}} \in (+C \cup +D) \cap (\overline{-C} \cup +D) \cap (+C \cup \overline{-D})$ by Definition 2. So $a^{\mathcal{I}} \in (+C \cap +D)$ where $C^{\mathcal{I}} = \langle +C, -C\rangle$ and $D^{\mathcal{I}} = \langle +D, -D\rangle$. Therefore, $\mathcal{I} \models_w (C \sqcap D)(a)$ by Definition 2.

  (b) (Inductive step) Assume that when the number of connectives and quantifiers in C is n, the theorem holds. We reduce axioms with $n + 1$ connectives and quantifiers into axioms with n connectives or quantifiers by equivalently eliminating one connective or quantifier. For instance, suppose $\phi = (C \sqcap (D \sqcup E))(a)$ where C, D, E are concepts and a is an individual. If $\mathcal{I} \models_s (C \sqcap (D \sqcup E))(a)$ then $a^{\mathcal{I}} \in +C$ and $a^{\mathcal{I}} \in +(D \sqcup E)$ where $C^{\mathcal{I}} = \langle +C, -C\rangle$ and $(D \sqcup E)^{\mathcal{I}} = \langle +(D \sqcup E), -(D \sqcup E)\rangle$, that is, $\mathcal{I} \models_s C(a)$ and $\mathcal{I} \models_s (D \sqcup E)(a)$. Thus, $\mathcal{I} \models_w C(a)$ and $\mathcal{I} \models_w D \sqcup E(a)$. Then $\mathcal{I} \models_w (C \sqcap (D \sqcup E))(a)$.

Therefore, if $\mathcal{I} \models_s C(a)$ then $\mathcal{I} \models_w C(a)$. □

This proposition shows that a strong model is also a weak model. As a result, the reasoning power of the strong satisfaction is no stronger than the weak satisfaction. In fact, the strong satisfaction is weaker than the weak satisfaction since the strong satisfaction is obtained by restricting some conditions of the weak satisfaction.

To see this, for instance, consider an ABox $\mathcal{A} = \{C(a), \neg C(a)\}$. Let $\mathcal{I}$ be a base interpretation such that $a^{\mathcal{I}} \in +C$ and $a^{\mathcal{I}} \in -C$ where $C^{\mathcal{I}} = \langle +C, -C \rangle$. Then $\mathcal{I} \models_w (C \sqcup D)(a)$ but $\mathcal{I} \not\models_s (C \sqcup D)(a)$.

In other words, the strong entailment does not satisfies the property of DI.

The following proposition states that the strong satisfaction $\models_s$ and the weak satisfaction $\models_w$ can preserve the property of intuitive equivalence under negation of concepts $\neg C$ and QC negation of concepts $\bar{C}$ respectively.

Proposition 4. *Given a QC GCI $C \sqsubseteq D$, for each base interpretation $\mathcal{I}$, we have*

(1) $\mathcal{I} \models_w C \sqsubseteq D$ *if and only if* $\mathcal{I} \models_w (\bar{C} \sqcup D)(a)$ *for any* $a \in \Delta$;
(2) $\mathcal{I} \models_s C \sqsubseteq D$ *if and only if* $\mathcal{I} \models_s (\neg C \sqcup D)(a)$ *for any* $a \in \Delta$.

Proof.

(1)

$$\begin{aligned} \mathcal{I} \models_w C \sqsubseteq D \quad &\Leftrightarrow \quad \text{for any } a,\ \text{if } a^{\mathcal{I}} \in +C \text{ then } a^{\mathcal{I}} \in +D; \\ &\Leftrightarrow \quad a^{\mathcal{I}} \notin +C \text{ or } a^{\mathcal{I}} \in +D; \\ &\Leftrightarrow \quad \mathcal{I} \models_w \bar{C}(a) \text{ or } \mathcal{I} \models_w D(a); \\ &\Leftrightarrow \quad \mathcal{I} \models_w (\bar{C} \sqcup D)(a); \end{aligned}$$

where $C^{\mathcal{I}} = \langle +C, -C \rangle$ and $D^{\mathcal{I}} = \langle +D, -D \rangle$.

(2)

$$\begin{aligned} &\mathcal{I} \models_s (\neg C \sqcup D)(a) \text{ for any } a \in \Delta; \\ \Leftrightarrow \quad &\text{for any } a \in \Delta,\ \big(a^{\mathcal{I}} \in +(\neg C) \text{ or } a^{\mathcal{I}} \in +D\big) \quad \text{and} \\ &\big(a^{\mathcal{I}} \in +C \text{ implies } a^{\mathcal{I}} \in +D\big) \text{ and } \big(a^{\mathcal{I}} \in -D \text{ implies } a^{\mathcal{I}} \in +(\neg C)\big); \\ \Leftrightarrow \quad &\overline{-C} \subseteq +D, +C \subseteq +D \text{ and } -D \subseteq -C,\ \text{where } C^{\mathcal{I}} = \langle +C, -C \rangle \text{ and } C^{\mathcal{I}} = \langle +D, -D \rangle; \\ \Leftrightarrow \quad &\mathcal{I} \models_s C \sqsubseteq D. \qquad \square \end{aligned}$$

Proposition 4 ensures that the problems about reasoning with TBoxes can be reduced to the problems about reasoning without TBoxes.

The above discussion (more comparison is shown in Table 2) shows that though the strong entailment $\models_s$ satisfies some useful reasoning rules, it is too weak. It implies none of the strong entailment and the weak entailment is a suitable paraconsistent semantics for QCDL.

For this reason, we introduce a novel consequence relation in terms of both the weak and the strong satisfaction relations. We define a QC entailment which is of the same form as classical entailment except that we use the strong satisfaction for the assumptions and weak satisfaction for the inferences. It is well known that the less assumptions are contained in the premise of an entailment, the more conclusions can be drawn. Based on this fact, the strong satisfaction is employed to make less assumptions in the premise in order to make QC semantics stronger. On the other hand, the weak satisfaction is employed to ensure the conclusion tolerating inconsistencies.

Definition 6. Let $\mathcal{K}$ be a QC KB and ϕ be a QC axiom. We call $\mathcal{K}$ quasi-classically entails (QC entails) ϕ, denoted $\mathcal{K} \models_Q \phi$, if for every base interpretation $\mathcal{I}$, $\mathcal{I} \models_s \mathcal{K}$ implies $\mathcal{I} \models_w \phi$. In this case, $\models_Q$ is called *QC entailment*.

Analogously, we can extendedly define $\mathcal{K}_1 \models_Q \mathcal{K}_2$ if $Mod^s(\mathcal{K}_1) \subseteq Mod^w(\mathcal{K}_2)$.

The reasoning problems w.r.t. the QC entailment is important, such as, instance checking ($\mathcal{K} \models_Q C(a)$) and subsumption checking ($\mathcal{K} \models_Q C \sqsubseteq D$).

A direct result is that the QC entailment satisfies the resolution among axioms of KBs.

Proposition 5. *For any B, C, E, $\{(B \sqcup C)(a), (\neg B \sqcup E)(a)\} \models_Q (C \sqcup E)(a)$.*

Proof. Let $\mathcal{A} = \{(B \sqcup C)(a), (\neg B \sqcup E)(a)\}$. Assume that $\mathcal{I}$ is a base interpretation of $\{(B \sqcup C)(a), (\neg B \sqcup E)(a)\}$, we have $\mathcal{I} \models_s (B \sqcup C)(a)$ and $\mathcal{I} \models_s (\neg B \sqcup E)(a)$. Thus, $a^{\mathcal{I}} \in (+B \cup +C) \cap (\overline{-B} \cup +C) \cap (+B \cup \overline{-C})$ and $a^{\mathcal{I}} \in (-B \cup +E) \cap (\overline{+B} \cup +E) \cap (-B \cup \overline{-E})$ by Definition 2 and Definition 4. Then $a^{\mathcal{I}} \in (+B \cup +C) \cap (-B \cup +E)$. We consider two cases in the following. (1) If $a^{\mathcal{I}} \in +B$ then $a^{\mathcal{I}} \in +E$. (2) If $a^{\mathcal{I}} \notin +B$ then $a^{\mathcal{I}} \in +C$. Therefore, $a^{\mathcal{I}} \in +C$ or $a^{\mathcal{I}} \in +E$. Thus, $\mathcal{I} \models_w (C \sqcup E)(a)$ by Definition 2 and Definition 4. So if $\mathcal{I} \models_s (B \sqcup C)(a)$ and $\mathcal{I} \models_s (\neg B \sqcup E)(a)$ then $\mathcal{I} \models_w (C \sqcup E)(a)$. Hence, $\mathcal{A} \models_Q (C \sqcup E)(a)$. $\square$

However, the resolution does not work in conclusions of the QC entailment.

For instance, $\{B(a), \neg B(a)\} \models_Q (B \sqcup C)(a)$ and $\{B(a), \neg B(a)\} \models_Q (\neg B \sqcup D)(a)$ while $\{B(a), \neg B(a)\} \not\models_Q (C \sqcup D)(a)$. Otherwise, it will inevitably cause explosive inference. That is, the resolution can be satisfied on axioms only occurring in KBs.

In this sense, $\models_Q$ preserves a the property of relevance (see Section 2.2 [34]).

By Proposition 5 and the second item of Proposition 4, it is not difficult to verify that $\models_s$ satisfies MP, MT and DS with maintaining relevance (more details are discussed in the next subsection).

Now, we define a new consistency called *quasi-classical consistency* (QC consistency, for short) in QCDLs. A QC KB $\mathcal{K}$ is *QC consistent* if there exists a strong model of $\mathcal{K}$, i.e., $Mod^s(\mathcal{K}) \neq \emptyset$. The strong model is similarly adopted to define QC models for propositional logic KBs [25].

For instance, let $\mathcal{A}_1 = \{A(a), \neg A(a)\}$ and $\mathcal{A}_2 = \{A(a), \overline{A}(a)\}$ be two QC ABoxes. Thus $\mathcal{A}_1$ is QC consistent (although $\mathcal{A}_1$ is inconsistent) while $\mathcal{A}_2$ is QC inconsistent. In a short, the QC consistency can be used to reserve inconsistency.

3.2. Paraconsistent reasoning using the QC entailment

In this subsection, we apply the QC entailment for paraconsistent reasoning with KBs and show that it can improve four-valued paraconsistent reasoning by enhancing the power of inference.

Firstly, we investigate some properties of the QC entailment between KBs and axioms (without QC negation).

To make the QC entailment $\models_Q$ always paraconsistent, i.e., we need to ensure that $Mod^s(\mathcal{K}) \neq \emptyset$ for any non-empty KB $\mathcal{K}$.

However, there exist three cases where no strong model satisfies as follows:

(1) an axiom $\bot(a)$ for some individual name a: for any strong interpretation $\mathcal{I}$ in a domain Δ, $\bot^{\mathcal{I}} = \langle \emptyset, \Delta^{\mathcal{I}} \rangle$, that is, $\mathcal{I} \not\models_s \bot(a)$ since $a^{\mathcal{I}} \notin \emptyset$ for any a.
(2) $\{a \doteq b, a \not\doteq b\}$ for some individual names a, b: for any strong interpretation $\mathcal{I}$, either $a^{\mathcal{I}} \neq b^{\mathcal{I}}$ or $a^{\mathcal{I}} \neq b^{\mathcal{I}}$, that is, either $\mathcal{I} \models_s a \doteq b$ or $\mathcal{I} \models_s a \not\doteq b$.
(3) $\{\leqslant n\, S(a), \geqslant (n+1)\, S(a)\}$ for some role S and some individual names a: for any strong interpretation $\mathcal{I}$, either $\sharp(\{y.(x, y) \in +S\}) \geqslant n+1$ or $\sharp(\{y.(x, y) \in +S\}) \leqslant n$ where $S^{\mathcal{I}} = \langle +S, -S \rangle$, that is, either $\mathcal{I} \models_s \leqslant n\, S(a)$ or $\mathcal{I} \models_s \geqslant (n+1)\, S(a)$.

In addition, a KB which infers the three cases (that is, one of consistent subsets of it can infer them) have no strong model.

For instance, a TBox $\mathcal{T} = \{\top \sqsubseteq \bot\}$ with a non-empty domain Δ has no any base interpretation $\mathcal{I}$ such that $\mathcal{I} \models_s \mathcal{T}$ because for any base interpretation $\mathcal{I}$, $\top^{\mathcal{I}} = \langle \Delta^{\mathcal{I}}, \emptyset \rangle$ and $\bot^{\mathcal{I}} = \langle \emptyset, \Delta^{\mathcal{I}} \rangle$ while $\Delta^{\mathcal{I}} \neq \emptyset$.

For instance, an ABox $\mathcal{A} = \{a \not\doteq b, \{a\}(b)\}$ with a non-empty domain Δ has no any base interpretation $\mathcal{I}$ such that $\mathcal{I} \models_s \mathcal{A}$ because for any base interpretation $\mathcal{I}$, $\{a\}^{\mathcal{I}} = \langle \{a^{\mathcal{I}}\}, N \rangle$ where $N \subseteq \Delta^{\mathcal{I}}$ and $a^{\mathcal{I}} \neq b^{\mathcal{I}}$ while $b^{\mathcal{I}} \in \{a^{\mathcal{I}}\}$, that is, $a^{\mathcal{I}} = b^{\mathcal{I}}$.

This problem is caused by the inherent feature of four-valued DLs [23].

Moreover, we investigate that the contradiction of $\{\leqslant n\, S(a), \geqslant (n+1)\, S(a)\}$ cannot be tolerated under our proposal QC semantics even under four-valued semantics since $+(\leqslant n\, S) \cap -(\leqslant n\, S) = \emptyset$ where $+(\leqslant n\, S) = \{x \mid \sharp(\{y.(x, y) \in +S\}) \leqslant n\}$ and $-(\leqslant n\, S) = \{x \mid \sharp(\{y.(x, y) \in +S\}) > n\}$ where $S^{\mathcal{I}} = \langle +S, -S \rangle$ for any base interpretation $\mathcal{I}$. In other words, for any instance a, either $a \in +(\leqslant n\, S)$ or $a \in -(\leqslant n\, S)$. The problem about tolerating inconsistency of number restrictions $\mathcal{N}$ under four-valued semantics also discussed in [21] is still open problem so far.

For all $\mathcal{N}$-free KBs, we adopt the a form of KBs so-called *satisfiable form* introduced by [23] by using the following two substitution rules:

(1) substituting $NA \sqcup \neg NA$ for $\top$ and $NA \sqcap \neg NA$ for $\bot$ where NA is a new concept name;
(2) substituting $\neg\{a\}(b)$ and $\neg\{b\}(a)$ for $a \not\doteq b$.

Let $SF(\mathcal{K})$ denote the satisfiable form of it.

Note that $SF(\mathcal{K})$ is equivalent to $\mathcal{K}$ under classical semantics. Let Δ be a domain. For any classical interpretation $\mathcal{I}_c$ in Δ, we have

(1) $(NA \sqcup \neg NA)^{\mathcal{I}_c} = NA^{\mathcal{I}_c} \cup (\neg NA)^{\mathcal{I}_c} = NA^{\mathcal{I}_c} \cup (\Delta^{\mathcal{I}_c} \setminus NA^{\mathcal{I}_c}) = \Delta^{\mathcal{I}_c} = \top^{\mathcal{I}_c}$ and $(NA \sqcap \neg NA)^{\mathcal{I}_c} = NA^{\mathcal{I}_c} \cap (\neg NA)^{\mathcal{I}_c} = NA^{\mathcal{I}_c} \cap (\Delta^{\mathcal{I}_c} \setminus NA^{\mathcal{I}_c}) = \emptyset = \bot^{\mathcal{I}_c}$.
(2) $\mathcal{I}_c \models \neg\{b\}(a)$ ($\mathcal{I}_c \models \neg\{a\}(b)$) if and only if $a^{\mathcal{I}_c} \neq b^{\mathcal{I}_c}$, that is, $\mathcal{I}_c \models a \not\doteq b$.

However, $SF(\mathcal{K})$ is no longer equivalent to $\mathcal{K}$ under the QC semantics.

For that TBox $\mathcal{T} = \{\top \sqsubseteq \bot\}$, $SF(\mathcal{T}) = \{(NA \sqcup \neg NA) \sqsubseteq (NA \sqcap \neg NA)\}$. Let Δ be a domain and $\mathcal{J}$ be a base interpretation such that $NA^{\mathcal{J}} = \langle \Delta^{\mathcal{J}}, \Delta^{\mathcal{J}} \rangle$. Thus $\mathcal{J} \models_s (NA \sqcup \neg NA) \sqsubseteq (NA \sqcap \neg NA)$ since $+(NA \sqcup \neg NA) = \Delta^{\mathcal{J}}$ and $+(NA \sqcap \neg NA) = \Delta^{\mathcal{J}}$.

For that ABox $\mathcal{A} = \{a \neq b, \{a\}(b)\}$, $SF(\mathcal{A}) = \{\neg\{a\}(b), \neg\{b\}(a), \{a\}(b)\}$. Let Δ be a domain and $\{a, b\} \subseteq \Delta$ and $\mathcal{J}$ be a base interpretation such that $\{a\}^{\mathcal{J}} = \langle\{a^{\mathcal{J}}\}, \{\Delta^{\mathcal{J}}\}\rangle$ and $\{b\}^{\mathcal{J}} = \langle\{b^{\mathcal{J}}\}, \{\Delta^{\mathcal{J}}\}\rangle$ where $a^{\mathcal{J}} = b^{\mathcal{J}}$. Thus $\mathcal{J} \models_s \neg\{a\}(b)$, $\mathcal{J} \models_s \neg\{b\}(a)$ and $\mathcal{J} \models_s \{a\}(b)$.

As a result, all KBs in satisfiable form have always strong models.

Proposition 6. *Let $\mathcal{K}$ be an $\mathcal{N}$-free KB. $Mod^s(SF(\mathcal{K})) \neq \emptyset$.*

To prove Proposition 6, inspiring from the trivial four-valued model of propositional logic where all atoms are assigned the truth of contradiction (i.e., "both true and false") (see [2]), given an $\mathcal{N}$-free $\mathcal{K}$, we introduce a strong interpretation called *trivial strong interpretation* $\mathcal{I}_0$ of $\mathcal{K}$ defined as follows: let Δ be a domain, σ an individual name in Δ and d_0 a concrete datatype in $\Delta_{\mathbf{D}}$ (we can also introduce $d_0, \ldots, d_m$ for $\Delta_0, \ldots, \Delta_m$ such that $d_i \in \Delta_i$ $(i = 1, \ldots, m)$. For simplify discussion, we mainly consider single $\Delta_{\mathbf{D}}$), for all $A, R, \{o\}, d, T$,

- $A^{\mathcal{I}_0} = \langle\{\sigma^{\mathcal{I}_0}\}, \{\sigma^{\mathcal{I}_0}\}\rangle$;
- $R^{\mathcal{I}_0} = \langle\{(\sigma^{\mathcal{I}_0}, \sigma^{\mathcal{I}_0})\}, \{(\sigma^{\mathcal{I}_0}, \sigma^{\mathcal{I}_0})\}\rangle$;
- $\{o\}^{\mathcal{I}_0} = \langle\{\sigma^{\mathcal{I}_0}\}, \{\sigma^{\mathcal{I}_0}\}\rangle$;
- $d^{\mathcal{I}_0} = \langle d_0, d_0\rangle$;
- $T^{\mathcal{I}_0} = \langle\{(\sigma^{\mathcal{I}_0}, d_0)\}, \{(\sigma^{\mathcal{I}_0}, d_0)\}\rangle$;
- $b^{\mathcal{I}_0} = \sigma^{\mathcal{I}_0}$ for all $b \in \Delta$;
- $d^{\mathcal{I}_0} = d_0$ for all $d \in \Delta_{\mathbf{D}}$;
- $\sigma^{\mathcal{I}_0} \in d_0$;
- complex concepts are defined following Definition 4.

In the trivial strong interpretation, all concepts in satisfiable form are identical.

Lemma 2. *Let $\mathcal{K}$ be an $\mathcal{N}$-free KB. For any concept C occurring in $SF(\mathcal{K})$, $C^{\mathcal{I}_0} = \langle\{\sigma^{\mathcal{I}_0}\}, \{\sigma^{\mathcal{I}_0}\}\rangle$.*

Proof. We prove this lemma by induction on the structure of C by Definition 4.

- (Base step) C is in form of A or $\{o\}$. By the definition above, we have $A^{\mathcal{I}_0} = \langle\{\sigma^{\mathcal{I}_0}\}, \{\sigma^{\mathcal{I}_0}\}\rangle$ and $\{o\}^{\mathcal{I}_0} = \langle\{\sigma^{\mathcal{I}_0}\}, \{\sigma^{\mathcal{I}_0}\}\rangle$.
- (Inductive step) Assume that for any C_i $(i = 1, 2)$, $C_i^{\mathcal{I}_0} = \langle\{\sigma^{\mathcal{I}_0}\}, \{\sigma^{\mathcal{I}_0}\}\rangle$.
  (1) C is in form of $\neg C_1$. Because $(\neg C_1)^{\mathcal{I}} = \langle -C_1, +C_1\rangle$ where $(\neg C_1) = \langle -C_1, +C_1\rangle$, this lemma holds for $\neg C_1$.
  (2) C is in form of $C_1 \sqcup C_2$. Because $C_i^{\mathcal{I}_0} = \langle\{\sigma^{\mathcal{I}_0}\}, \{\sigma^{\mathcal{I}_0}\}\rangle$ $(i = 1, 2)$, $(C_1 \sqcup C_2)^{\mathcal{I}_0} = \langle\{\sigma^{\mathcal{I}_0}\}, \{\sigma^{\mathcal{I}_0}\}\rangle$ by Definition 4.
  (3) C is in form of $C_1 \sqcap C_2$. Because $C_i^{\mathcal{I}_0} = \langle\{\sigma^{\mathcal{I}_0}\}, \{\sigma^{\mathcal{I}_0}\}\rangle$ $(i = 1, 2)$, $(C_1 \sqcup C_2)^{\mathcal{I}_0} = \langle\{\sigma^{\mathcal{I}_0}\}, \{\sigma^{\mathcal{I}_0}\}\rangle$ by Definition 4.
  (4) C is in form of $\forall R.C_1$. Because $C_1^{\mathcal{I}_0} = \langle\{\sigma^{\mathcal{I}_0}\}, \{\sigma^{\mathcal{I}_0}\}\rangle$, $(\forall R.C_1)^{\mathcal{I}_0} = \langle\{\sigma^{\mathcal{I}_0}\}, \{\sigma^{\mathcal{I}_0}\}\rangle$ since $(\sigma^{\mathcal{I}_0}, \sigma^{\mathcal{I}_0}) \in +R$ where $R^{\mathcal{I}_0} = \langle +R, -R\rangle$ by Definition 4.
  (5) C is in form of $\exists R.C_1$. Because $C_1^{\mathcal{I}_0} = \langle\{\sigma^{\mathcal{I}_0}\}, \{\sigma^{\mathcal{I}_0}\}\rangle$. $(\exists R.C_1)^{\mathcal{I}_0} = \langle\{\sigma^{\mathcal{I}_0}\}, \{\sigma^{\mathcal{I}_0}\}\rangle$ since $(\sigma^{\mathcal{I}_0}, \sigma^{\mathcal{I}_0}) \in +R$ where $R^{\mathcal{I}_0} = \langle +R, -R\rangle$ by Definition 4.
  (6) C is in form of $\forall T.d$. Because $d^{\mathcal{I}_0} = \langle d_0, d_0\rangle$ and $(\sigma^{\mathcal{I}_0}, d_0) \in +T$ where $T^{\mathcal{I}_0} = \langle +T, -T\rangle$, $(\forall T.d)^{\mathcal{I}_0} = \langle\{\sigma^{\mathcal{I}_0}\}, \{\sigma^{\mathcal{I}_0}\}\rangle$ by Definition 4.
  (7) C is in form of $\exists T.d$. Because $d^{\mathcal{I}_0} = \langle d_0, d_0\rangle$ and $(\sigma^{\mathcal{I}_0}, d_0) \in +T$ where $T^{\mathcal{I}_0} = \langle +T, -T\rangle$, $(\exists T.d)^{\mathcal{I}_0} = \langle\{\sigma^{\mathcal{I}_0}\}, \{\sigma^{\mathcal{I}_0}\}\rangle$ by Definition 4. $\square$

Next, we show that the trivial strong interpretation $\mathcal{I}_0$ is a strong model of $SF(\mathcal{K})$ for any $\mathcal{K}$ in $\mathcal{SHOIN}(\mathbf{D})$. In this sense, we called *trivial strong model*.

Proof of Proposition 6. By Proposition 28 in [24] and the previous TBox $\mathcal{T} = \{\top \sqsubseteq \bot\}$, we only need to show the other case: let $\mathcal{A} = \{\bot(a)\}$, $SF(\mathcal{A}) = \{(NA \sqcap \neg NA)(a)\}$. Let $\mathcal{I}$ be a base interpretation such that $NA^{\mathcal{I}} = \langle\{a^{\mathcal{I}}\}, \{a^{\mathcal{I}}\}\rangle$ where $a \in \Delta$. Thus $\mathcal{I} \models_s (NA \sqcap \neg NA)(a)$.

Now, we mainly show that for any $\mathcal{N}$-free KB $\mathcal{K}$ without $\top$ or $\bot$, has $\mathcal{I}_0$ is a strong model of $SF(\mathcal{K})$ by Definition 5. That is, for all $\varphi \in \mathcal{K}$, $\mathcal{I}_0 \models_s \varphi$.

(1) φ is in form of $C(a)$. By Lemma 2, because $C^{\mathcal{I}_0} = \langle\{\sigma^{\mathcal{I}_0}\}, \{\sigma^{\mathcal{I}_0}\}\rangle$, $a^{\mathcal{I}_0} \in \{\sigma^{\mathcal{I}_0}\}$. Therefore, $\mathcal{I}_0 \models_s C(a)$.
(2) φ is in form of $d(a)$. Because $a^{\mathcal{I}_0} = \sigma^{\mathcal{I}_0}$ and $d^{\mathcal{I}_0} = d_0$, $a^{\mathcal{I}_0} \in d^{\mathcal{I}_0}$ since $\sigma^{\mathcal{I}_0} \in d_0$. Therefore, $\mathcal{I}_0 \models_s d(a)$.
(3) φ is in form of $R(a, b)$. Because $(a^{\mathcal{I}_0}) = \sigma^{\mathcal{I}_0}$ and $b^{\mathcal{I}_0} = \sigma^{\mathcal{I}_0}$, $+R = \{(\sigma^{\mathcal{I}_0}, \sigma^{\mathcal{I}_0})\}$ where $R^{\mathcal{I}_0} = \langle +R, -R\rangle$, $(a^{\mathcal{I}_0}, b^{\mathcal{I}_0}) \in +R$. Therefore, $\mathcal{I}_0 \models_s R(a, b)$.
(4) φ is in form of $T(a, d)$. Because $a^{\mathcal{I}_0} = \sigma^{\mathcal{I}_0}$, $d^{\mathcal{I}_0} = d_0$ and $+T = \{(\sigma^{\mathcal{I}_0}, d_0)\}$ where $T^{\mathcal{I}_0} = \langle +T, -T\rangle$, $(a^{\mathcal{I}_0}, d^{\mathcal{I}_0}) \in +T$. Therefore, $\mathcal{I}_0 \models_s T(a, d)$.

(5) φ is in form of $Trans(R)$. Because $+R = \{(\sigma^{\mathcal{I}_0}, \sigma^{\mathcal{I}_0})\}$ and $+R^{tc} = \{(\sigma^{\mathcal{I}_0}, \sigma^{\mathcal{I}_0})\}$ where $R^{\mathcal{I}_0} = \langle +R, -R\rangle$, $+R \subseteq +R^{tc}$. Therefore, $\mathcal{I}_0 \models_s Trans(R)$.
(6) φ is in form of $a \doteq b$. Because $a^{\mathcal{I}_0} = \sigma^{\mathcal{I}_0}$ and $b^{\mathcal{I}_0} = \sigma^{\mathcal{I}_0}$, $a^{\mathcal{I}_0} = b^{\mathcal{I}_0}$. Therefore, $\mathcal{I}_0 \models_s a \doteq b$.
(7) φ is in form of $C_1 \sqsubseteq C_2$. By Lemma 2, we have $C_i^{\mathcal{I}_0} = \langle\{\sigma^{\mathcal{I}_0}\}, \{\sigma^{\mathcal{I}_0}\}\rangle$ $(i = 1, 2)$. That is, $+C_i = \{\sigma^{\mathcal{I}_0}\}$ and $-C_i = \{\sigma^{\mathcal{I}_0}\}$ $(i = 1, 2)$. Then
 (i) because $\Delta^{\mathcal{I}} \setminus \{\sigma^{\mathcal{I}_0}\} = \emptyset \subseteq \{\sigma^{\mathcal{I}_0}\}$, we have $\overline{-C_1} \subseteq +C_2$ since $\Delta^{\mathcal{I}} = \{\sigma^{\mathcal{I}_0}\}$;
 (ii) because $\{\sigma^{\mathcal{I}_0}\} \subseteq \{\sigma^{\mathcal{I}_0}\}$, we have $+C_1 \subseteq +C_2$ and $-C_2 \subseteq -C_1$.
 Therefore, $\mathcal{I}_0 \models_s C_1 \sqsubseteq C_2$ by Definition 5.
(8) φ is in form of $d_1 \sqsubseteq d_2$. By Lemma 2, we have $d_i^{\mathcal{I}_0} = \langle d_0, d_0\rangle$ $(i = 1, 2)$. That is, $+d_i = d_0$ and $-d_i = d_0$ $(i = 1, 2)$. Then
 (i) because $\Delta_{\mathbf{D}} \setminus d_0 = \emptyset \subseteq d_0$, we have $\overline{-d_1} \subseteq +d_2$ since $\Delta_{\mathbf{D}} = d_0$;
 (ii) because $d_0 \subseteq d_0$, we have $+d_1 \subseteq +d_2$ and $-d_2 \subseteq -d_1$.
 Therefore, $\mathcal{I}_0 \models_s d_1 \sqsubseteq d_2$ by Definition 5.
(9) φ is in form of $R_1 \sqsubseteq R_2$. By Lemma 2, we have $R_i^{\mathcal{I}_0} = \langle\{(\sigma^{\mathcal{I}_0}, \sigma^{\mathcal{I}_0})\}, \{(\sigma^{\mathcal{I}_0}, \sigma^{\mathcal{I}_0})\}\rangle$ $(i = 1, 2)$. That is, $+R_i = \{(\sigma^{\mathcal{I}_0}, \sigma^{\mathcal{I}_0})\}$ and $-R_i = \{(\sigma^{\mathcal{I}_0}, \sigma^{\mathcal{I}_0})\}$ $(i = 1, 2)$. Then
 (i) because $\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \setminus \{(\sigma^{\mathcal{I}_0}, \sigma^{\mathcal{I}_0})\} = \emptyset \subseteq \{(\sigma^{\mathcal{I}_0}, \sigma^{\mathcal{I}_0})\}$, we have $\overline{-R_1} \subseteq +R_2$ since $\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} = \{(\sigma^{\mathcal{I}_0}, \sigma^{\mathcal{I}_0})\}$;
 (ii) because $\{(\sigma^{\mathcal{I}_0}, \sigma^{\mathcal{I}_0})\} \subseteq \{(\sigma^{\mathcal{I}_0}, \sigma^{\mathcal{I}_0})\}$, we have $+R_1 \subseteq +R_2$ and $-R_2 \subseteq -R_1$.
 Therefore, $\mathcal{I}_0 \models_s R_1 \sqsubseteq R_2$ by Definition 5. Analogously, we can conclude that $\mathcal{I}_0 \models_s T_1 \sqsubseteq T_2$.

In a short, $\mathcal{I}_0 \models_s \varphi$ for all $\varphi \in SF(\mathcal{K})$ for any $\mathcal{N}$-free KB $\mathcal{K}$ in $\mathcal{SHOIN}(\mathbf{D})$. $\square$

We also notice that the satisfiable form of some tautologies can be no always satisfied by all weak interpretations since the weak satisfaction inherits the four-valued satisfaction.

For instance, let $\Delta = \{a, b, c\}$ and $\mathcal{I}$ be a weak interpretation such that $NA^{\mathcal{I}} = \langle\{a^{\mathcal{I}}\}, \{b^{\mathcal{I}}\}\rangle$. Thus $\mathcal{I}$ does not satisfy $(NA \sqcup \neg NA)(c)$ since $(NA \sqcup \neg NA)^{\mathcal{I}} = \langle\{a^{\mathcal{I}}, b^{\mathcal{I}}\}, \emptyset\rangle$.

Besides, in some engineering point of view, tautologies do not provide any useful information to users [19,4]. In other words, the QC entailment does not satisfy EM, that is, $\emptyset \models_Q (C \sqcup \neg C)(a)$.

Based on discussion above, we mainly consider all $\mathcal{N}$-free KBs in satisfiable form, where the QC entailment is always paraconsistent.

Proposition 7. *Let $\mathcal{K}$ be an $\mathcal{N}$-free KB. There exists always an axiom φ such that $SF(\mathcal{K}) \not\models_Q \varphi$.*

Proof. Let $\Sigma(\mathcal{K})$ be a set of all concept names, role names and datatypes in $\mathcal{K}$. Without loss of generality, we assume that φ is $A(a)$ where $A \notin \Sigma(\mathcal{K})$. Let $\mathcal{I}_0$ be a strong model of $SF(\mathcal{K})$ by Proposition 6. We define a new interpretation $\mathcal{I}_{new}$ as follows: (1) $X^{\mathcal{I}_{new}} = \langle\emptyset, \{\sigma^{\mathcal{I}_0}\}\rangle$ if $X = A_{new}$; and (2) $X^{\mathcal{I}_{new}} = X^{\mathcal{I}}$ otherwise. Then $\mathcal{I}_{new}$ is also a strong model of $SF(\mathcal{K})$ while $\mathcal{I}_{new}$ is not a weak model of $A(a)$. Therefore, $SF(\mathcal{K}) \not\models_Q A(a)$. That is, $SF(\mathcal{K}) \not\models_Q \varphi$. $\square$

We notice that if the inconsistency is not caused by number restrictions ($\mathcal{N}$) then the paraconsistency of QCDLs holds in KBs. Indeed, we found that most of practical ontologies do not bring the inconsistency of number restrictions (i.e., Pellet [37] and TONES Ontology Repository [41]).

In general, the paraconsistency of QCDLs fails in general QC KBs.

For instance, $\{A(a), \overline{A}(a)\} \models_Q \phi$ for any axiom ϕ since there exists no strong model of $\{A(a), \overline{A}(a)\}$.

The following proposition shows that $\models_Q$ satisfies MP, MT and DS because of holding the property of resolution of the QC entailment.

Proposition 8. *Let C, D be two DL concepts. The followings hold.*

(1) **MP**: $\{C(a), C \sqsubseteq D\} \models_Q D(a)$;
(2) **MT**: $\{\neg D(a), C \sqsubseteq D\} \models_Q C(a)$;
(3) **DS**: $\{\neg C(a), (C \sqcup D)(a)\} \models_Q D(a)$.

Proof. By Proposition 4, for any base interpretation $\mathcal{I}$, if $\mathcal{I} \models_s C \sqsubseteq D$ then $\mathcal{I} \models_s (\neg C \sqcup \neg D)(a)$. Thus items (1) and (2) can be reduced into item (3) Then, by Proposition 5, $\mathcal{I} \models_s D(a)$, that is, item (3) holds. $\square$

We can obtain much more conclusions from inconsistent KBs via the QC entailment.

For instance, let $\mathcal{A} = \{\neg Student(Wade), Student \sqcup Staff(Wade)\}$ be an ABox. Thus $\mathcal{A} \models_Q Staff(Wade)$ by using the property of DS.

In addition, similar to the four-valued entailment, the QC entailment also satisfies DI in the following proposition.

Proposition 9. *Let C, D be two DL concepts. $\{C(a)\} \models_Q (C \sqcup D)(a)$.*

Table 2
Comparisons among four entailments.

Property	Entailment					
	$\models_4$			$\models_w$	$\models_s$	$\models_Q$
	$\mapsto$	$\sqsubset$	$\rightarrow$			
MP	no	yes	yes	yes	yes	yes
MT	no	no	yes	no	yes	yes
DS	no	no	no	no	yes	yes
DI	yes	yes	yes	yes	no	yes
IE	no	no	no	yes*	yes	yes
resolution	no	no	no	no	yes	yes
transitivity	yes	yes	yes	yes	yes	no
EM	no	no	no	no	no	no

Proof. For any base interpretation $\mathcal{I}$, $\mathcal{I} \models_s C(a)$ if and only if $a^{\mathcal{I}} \in +C$ where $C^{\mathcal{I}} = \langle +C, -C \rangle$. Thus $a^{\mathcal{I}} \in +C \cup +D$ where $D^{\mathcal{I}} = \langle +D, -D \rangle$. Then $\mathcal{I} \models_w (C \sqcup D)(a)$. Therefore, $\{C(a)\} \models_Q (C \sqcup D)(a)$. $\square$

For general KBs, the transitivity does not satisfied by the QC entailment.

For instance, let $\mathcal{A}_1 = \{A(a), \neg A(a)\}$, $\mathcal{A}_2 = \{A(a), (\neg A \sqcup B)(a)\}$ and $\mathcal{A}_3 = \{(B \sqcup C)(a)\}$. Thus $\mathcal{A}_1 \models_Q \mathcal{A}_2$ and $\mathcal{A}_2 \models_Q \mathcal{A}_3$ while $\mathcal{A}_1 \not\models_Q \mathcal{A}_3$ obviously.

A more detailed comparison can be shown in Table 2 where yes* means that this property w.r.t. QC negation is still satisfied by the weak entailment (see the first item of Proposition 4).

In the end of this section, we show that some paraconsistent reasoning problems via the QC entailment can be reduced into the QC consistency problem.

Firstly, two important kinds of QC entailment problems (instance checking and subsumption checking) can be reduced into the QC consistency problem.

Proposition 10. *Let $\mathcal{T}$ be a terminology, $\mathcal{R}$ a role hierarchy, $\mathcal{A}$ an ABox and C, D concepts. For any base interpretation $\mathcal{I}$, we interpret $U^{\mathcal{I}} = \langle \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}, \emptyset \rangle$. Then*

(1) *$(\mathcal{T}, \mathcal{R}, \mathcal{A}) \models_Q C(a)$ if and only if $(\mathcal{T}, \mathcal{R}, \mathcal{A} \cup \{\overline{C}(a)\})$ is QC inconsistent w.r.t. $\mathcal{R}_U$;*
(2) *$(\mathcal{T}, \mathcal{R}, \emptyset) \models_Q C \sqsubseteq D$ if and only if $(\mathcal{T}, \mathcal{R}, \{(C \sqcap \overline{D})(\iota)\})$ is QC inconsistent w.r.t. $\mathcal{R}_U$ for some new individual $\iota \in \Delta$.*

Proof.

(1) ($\Leftarrow$) Suppose $\mathcal{K} \not\models_Q C(a)$, by Definition 6 there exists a base interpretation $\mathcal{I}$ such that $\mathcal{I} \models_s \mathcal{K}$ but $\mathcal{I} \not\models_w C(a)$, i.e., $\mathcal{I} \not\models_s C(a)$ by Proposition 3. Since $\mathcal{I} \models_s \mathcal{K}$, $\mathcal{I} \models_s \mathcal{K} \cup \{\overline{C}(a)\}$, that is, $\mathcal{I}$ is a QC strong model of $\mathcal{K} \cup \{\overline{C}(a)\}$ which contradicts the premise that $\mathcal{K} \cup \{\overline{C}(a)\}$ is QC inconsistent by the definition of QC inconsistency.
($\Rightarrow$) Suppose $\mathcal{K} \cup \{\overline{C}(a)\}$ is QC consistent. Thus, there exists a base interpretation $\mathcal{J}$ such that $\mathcal{J} \models_s \mathcal{K} \cup \{\overline{C}(b)\}$ by the definition of QC inconsistency. Then $\mathcal{J} \models_s \mathcal{K}$ and $\mathcal{J} \models_s \overline{C}(b)$. That is, $a^{\mathcal{J}} \notin +C$ where $C^{\mathcal{J}} = \langle +C, -C \rangle$. Then $\mathcal{J} \not\models_w C(a)$, i.e., $\mathcal{J} \models_w \overline{C}(a)$ by the definition of the complement of concepts. Therefore, $\mathcal{K} \not\models_Q C(a)$ which contradicts the premise that $\mathcal{K} \models_Q C(a)$.
(2) ($\Leftarrow$) Supposed that $\mathcal{T} \not\models_Q C \sqsubseteq D$, by Definition 6 there exists an interpretation $\mathcal{I}$ such that $\mathcal{I} \models_s \mathcal{T}$ but $\mathcal{I} \not\models_w C \sqsubseteq D$, i.e., $\mathcal{I} \not\models_s C \sqsubseteq D$ by Proposition 3. Thus, there exists an individual a such that if $\mathcal{I} \models_s C(a)$ then $\mathcal{I} \models_s \overline{D}(a)$ by Proposition 3. Then $\mathcal{I} \models_s C \sqcap \overline{D}(a)$. Since $\mathcal{I} \models_s \mathcal{K}$, $\mathcal{I} \models_s (\mathcal{T}, \{(C \sqcap \overline{D})(a)\})$, that is, $\mathcal{I}$ is a QC strong model of $(\mathcal{T}, \{(C \sqcap \overline{D})(a)\})$ which contradicts the premise that for any individual a, $(\mathcal{T}, \{(C \sqcap \overline{D})(a)\})$ is QC inconsistent by the definition of QC inconsistency.
($\Rightarrow$) Suppose there exists an individual ι such that $\mathcal{K} \cup \{(C \sqcap \overline{D})(\iota)\}$ is QC consistent. Thus, there exists a base interpretation $\mathcal{J}$ such that $\mathcal{J} \models_s \mathcal{T} \cup \{(C \sqcap \overline{D})(\iota)\}$ by the definition of QC inconsistency. Then $\mathcal{J} \models_s \mathcal{T}$, $\mathcal{J} \models_s C(\iota)$ and $\mathcal{J} \models_s \overline{D}(\iota)$. That is, $b^{\mathcal{J}} \in +C$ and $\iota^{\mathcal{J}} \notin +D$ where $C^{\mathcal{J}} = \langle +C, -C \rangle$ and $D^{\mathcal{J}} = \langle +D, -D \rangle$. Then $\mathcal{J} \not\models_w C \sqsubseteq D$, i.e., $\mathcal{J} \models_w (C \sqcap \overline{D})(\iota)$ for any individual ι by Proposition 4. Therefore, $\mathcal{T} \not\models_Q C \sqsubseteq D$ which contradicts the premise that $\mathcal{T} \models_Q C \sqsubseteq D$.

Here "$\Leftarrow$" denotes the "only if" direction and "$\Rightarrow$" denotes the "if" direction respectively in this paper. $\square$

Additionally, we define a new satisfiable, called *quasi-classical satisfiable* (QC satisfiable). A concept C is QC satisfiable w.r.t. $\mathcal{T}$ and $\mathcal{R}$ if there exists some strong model $\mathcal{I}$ of $\mathcal{T}$ and $\mathcal{R}$ such that $+C \neq \emptyset$ where $C^{\mathcal{I}} = \langle +C, -C \rangle$; and QC unsatisfiable, otherwise. We analogously conclude that C is QC unsatisfiable w.r.t. $\mathcal{T}$ and $\mathcal{R}$ if and only if $(\mathcal{T}, \mathcal{R}, \{\overline{C}(\iota)\})$ is QC inconsistent w.r.t. $\mathcal{T}$ and $\mathcal{R}$ for some new individual $\iota \in \Delta$.

As a result, the QC satisfiability checking can be reduced to the QC consistency checking.

Table 3
Weak transformation rules for $\mathcal{SHOIN}(\mathbf{D})$.

Syntax	Weak transformation $\mathcal{W}(\cdot)$
$\top$	$NA \sqcup \neg NA$
$\bot$	$NA \sqcap \neg NA$
$\neg\top$	$NA \sqcap \neg NA$
$\neg\bot$	$NA \sqcup \neg NA$
X $(X \in \{A, R, S, T, d\})$	X^+
$\neg X$ $(X \in \{A, R, S, T, d\})$	X^-
$\{o\}$	$\{o\}$
$\neg\{o\}$	A_o: a new concept name
$\overline{C}$	$\neg\mathcal{W}(C)$
$\overline{\overline{C}}$	$\mathcal{W}(C)$
$\neg\neg C$	$\mathcal{W}(C)$
$C \sqcap D$	$\mathcal{W}(C) \sqcap \mathcal{W}(D)$
$\neg(C \sqcap D)$	$\mathcal{W}(\neg C) \sqcup \mathcal{W}(\neg D)$
$C \sqcup D$	$\mathcal{W}(C) \sqcup \mathcal{W}(D)$
$\neg(C \sqcup D)$	$\mathcal{W}(\neg C) \sqcap \mathcal{W}(\neg D)$
$\exists R.C$	$\exists\mathcal{W}(R).\mathcal{W}(C)$
$\forall R.C$	$\forall\mathcal{W}(R).\mathcal{W}(C)$
$\neg(\exists R.C)$	$\forall\mathcal{W}(R).\mathcal{W}(\neg C)$
$\neg(\forall R.C)$	$\exists\mathcal{W}(R).\mathcal{W}(\neg C)$
$\exists T.d$	$\exists\mathcal{W}(T).\mathcal{W}(d)$
$\forall T.d$	$\forall\mathcal{W}(T).\mathcal{W}(d)$
$\neg(\exists T.d)$	$\forall\mathcal{W}(T).\mathcal{W}(\neg d)$
$\neg(\forall T.d)$	$\exists\mathcal{W}(T).\mathcal{W}(\neg d)$
$\geqslant n\, S$	$\geqslant n\, \mathcal{W}(S)$
$\leqslant n\, S$	$\leqslant n\, \mathcal{W}(S)$
$\neg(\geqslant n\, S)$	$\leqslant (n-1)\, \mathcal{W}(S)$
$\neg(\leqslant n\, S)$	$\geqslant (n+1)\, \mathcal{W}(S)$
$C(a)$	$\mathcal{W}(C)(a)$
$C \sqsubseteq D$	$\mathcal{W}(C) \sqsubseteq \mathcal{W}(D)$
$R(a, b)$	$\mathcal{W}(R)(a, b)$
$R_1 \sqsubseteq R_2$	$\mathcal{W}(R_1) \sqsubseteq \mathcal{W}(R_2)$
$Trans(R)$	$Trans(\mathcal{W}(R))$
$a \neq b$	$a \neq b$
$a \doteq b$	$a \doteq b$

4. Reducing QC-$\mathcal{SHOIN}(\mathbf{D})$ to $\mathcal{SHOIN}(\mathbf{D})$

In the previous section, we introduced the QC semantics and some properties of this proposal semantics and discussed the QC entailment problems (as important tasks). In this section, we study some algorithm for reasoning with QC-$\mathcal{SHOIN}(\mathbf{D})$.

As we all known, building an OWL reasoner is very complicated. We do not want start from scratch. The motivation of candidate algorithm is reusing existing OWL reasoners. Inspiring from the technique of transformation in QC logic in which the QC inference problem can be reduced into the classical inference problem [25], we will develop a syntactic transformation that reduces the QC entailment problems in QC-$\mathcal{SHOIN}(\mathbf{D})$ to the classical entailment problems in $\mathcal{SHOIN}(\mathbf{D})$. As a result, we are able to build our reasoner named PROSE based on off-the-shelf OWL reasoners. Indeed, the similar technique is adopted in four-valued DLs for reasoning [23].

Our proposal transformation for QC-$\mathcal{SHOIN}(\mathbf{D})$ contains two transformations: *weak transformation* and *strong transformation*. The weak transformation is defined in Table 3 where NA is a new concept name.

The basic idea of the weak transformation is transforming the negation of a concept name (a role name, a nominal and a datatype) to a new concept name (a new role name, a new concept name and a new datatype) to violate the connection between a concept name and its negation. Thus, the conflict caused by a concept and its negation can be tolerated in reasoning.

The strong transformation $\mathcal{S}(\cdot)$ which is identical with the weak transformation $\mathcal{W}(\cdot)$ except for disjunctions and the negation of conjunctions and GCIs defined as follows: when $X_i \in \{C_i, R_i\}$ $(i = 1, 2)$,

$$\mathcal{S}\big(\neg(C \sqcap D)\big) = \mathcal{S}(\neg C \sqcup \neg D); \tag{12}$$

$$\mathcal{S}(C \sqcup D) = \big(\mathcal{S}(C) \sqcup \mathcal{S}(D)\big) \sqcap \big(\neg\mathcal{S}(\neg C) \sqcup \mathcal{S}(D)\big) \sqcap \big(\mathcal{S}(C) \sqcup \neg\mathcal{S}(\neg D)\big); \tag{13}$$

$$\mathcal{S}(X_1 \sqsubseteq X_2) = \big\{\mathcal{S}(X_1) \sqsubseteq \mathcal{S}(X_2), \mathcal{S}(\neg X_2) \sqsubseteq \mathcal{S}(\neg X_1), \neg\mathcal{S}(\neg X_1) \sqsubseteq X_2\big\}. \tag{14}$$

In weak transformation and strong transformation, (1) each object X in QCDLs, X and $\neg X$ can transformed into two independent concept names X^+ and X^- respectively. It follows from the principle of QCDLs that X and $\neg X$ are no longer taken as two opposite objects in QCDLs (where $X \in \{A, R, T, d, \{o\}\}$). (2) The QC negation of concepts $\overline{C}$ is transformed into

$\neg C$. It follows from that fact that A and $\overline{A}$ are taken as two opposite concept names in QCDLs. Therefore, all QC axioms can be transformed into axioms.

Given a QC KB $\mathcal{K}$, $\mathcal{X}(\mathcal{K})$ denotes the new KB whose axioms are transformed by using $\mathcal{X}$ where $\mathcal{X}$ is the placeholder of $\mathcal{W}$ and $\mathcal{S}$.

The following proposition shows that the weak transformation $\mathcal{W}(\cdot)$ and the strong transformation $\mathcal{S}(\cdot)$ can capture exactly the weak satisfaction $\models_w$ and the strong satisfaction $\models_s$ respectively.

Proposition 11. *For any base interpretation $\mathcal{I}$, there exists a classical interpretation $\mathcal{I}_c$ such that for any QC KB $\mathcal{K}$, we have*
(1) $\mathcal{I} \models_w \mathcal{K} \Leftrightarrow \mathcal{I}_c \models \mathcal{W}(\mathcal{K})$;
(2) $\mathcal{I} \models_s \mathcal{K} \Leftrightarrow \mathcal{I}_c \models \mathcal{S}(\mathcal{K})$.

The first item of this proposition can be analogously proven in four-valued transformation (see the proof of Proposition 31 in [24]) by restricting inclusion to be interpreted as internal inclusion. Here we omit its details.

Next, we will inductively prove the second item of this proposition.

Proof of item (2) of Proposition 11. Given a base interpretation $\mathcal{I}$, we define the corresponding classical interpretation $\mathcal{I}_c$ (called an *induced interpretation* [24]) as follows:

- $\Delta^{\mathcal{I}_c} = \Delta^{\mathcal{I}}$;
- $a^{\mathcal{I}_c} = a^{\mathcal{I}}$;
- $\top^{\mathcal{I}_c} = \Delta^{\mathcal{I}}$;
- $\bot^{\mathcal{I}_c} = \emptyset$;
- $(A^+)^{\mathcal{I}_c} = +A$ and $(A^-)^{\mathcal{I}_c} = -A$ if $A^{\mathcal{I}} = \langle +A, -A\rangle$;
- $(R^+)^{\mathcal{I}_c} = +R$, if $R^{\mathcal{I}} = \langle +R, -R\rangle$;
- $(T^+)^{\mathcal{I}_c} = +T$, if $T^{\mathcal{I}} = \langle +T, -T\rangle$;
- $(d^+)^{\mathcal{I}_c} = +d$ and $(d^-)^{\mathcal{I}_c} = -d$ if $d^{\mathcal{I}} = \langle +d, -d\rangle$;
- $\{o\}^{\mathcal{I}_c} = \{o^{\mathcal{I}}\}$ and $(A_o)^{\mathcal{I}_c} = N$, if $\{o\}^{\mathcal{I}} = \langle \{o^{\mathcal{I}}\}, N\rangle$;
- complex concepts are defined following the definition of classical interpretations in DLs shown in Table 1.

We only need to show the following claim.

Claim. *For any φ in QC-$\mathcal{SHOIN}(\mathbf{D})$, $\mathcal{I} \models_s \varphi \Leftrightarrow \mathcal{I}_c \models \mathcal{S}(\varphi)$.*

Next, we prove this claim in considering eight kinds of axioms.

(1) φ is in form of $R(a,b)$: $\mathcal{I} \models_s R(a,b) \Leftrightarrow (a^{\mathcal{I}}, b^{\mathcal{I}}) \in +R$ where $R^{\mathcal{I}} = \langle +R, -R\rangle \Leftrightarrow (a^{\mathcal{I}_c}, b^{\mathcal{I}_c}) \in (R^+)^{\mathcal{I}_c} \Leftrightarrow \mathcal{I}_c \models R^+(a,b) \Leftrightarrow \mathcal{I}_c \models \mathcal{S}(R)(a,b)$.
(2) φ is in form of $T(a,b)$: $\mathcal{I} \models_s T(a,b) \Leftrightarrow (a^{\mathcal{I}}, b^{\mathcal{I}}) \in +T$ where $T^{\mathcal{I}} = \langle +T, -T\rangle \Leftrightarrow (a^{\mathcal{I}_c}, b^{\mathcal{I}_c}) \in (T^+)^{\mathcal{I}_c} \Leftrightarrow \mathcal{I}_c \models T^+(a,b) \Leftrightarrow \mathcal{I}_c \models \mathcal{S}(T)(a,b)$.
(3) φ is in form of $Trans(R)$: $\mathcal{I} \models_s Trans(R) \Leftrightarrow +R \subseteq (+R)^{tc}$ where $R^{\mathcal{I}} = \langle +R, -R\rangle \Leftrightarrow (R^+)^{\mathcal{I}_c} \subseteq ((R^+)^{\mathcal{I}_c})^{tc} \Leftrightarrow \mathcal{I}_c \models Trans(R^+) \Leftrightarrow \mathcal{I}_c \models Trans(\mathcal{S}(R))$.
(4) φ is in form of $R_1 \sqsubseteq R_2$: $\mathcal{I} \models_s R_1 \sqsubseteq R_2 \Leftrightarrow \overline{-R_1} \subseteq +R_2, +R_1 \subseteq +R_2$ and $-R_2 \subseteq -R_1$ where $R_i^{\mathcal{I}} = \langle +R_i, -R_i\rangle$ and $-R_i = \overline{+R_i}$ $(i=1,2) \Leftrightarrow +R_1 \subseteq +R_2 \Leftrightarrow (R_1^+)^{\mathcal{I}_c} \subseteq (R_2^+)^{\mathcal{I}_c} \Leftrightarrow \mathcal{I}_c \models R_1^+ \sqsubseteq R_2^+ \Leftrightarrow \mathcal{I}_c \models \mathcal{S}(R_1) \sqsubseteq \mathcal{S}(R_2)$.
(5) φ is in form of $a \doteq b$: $\mathcal{I} \models_s a \doteq b \Leftrightarrow a^{\mathcal{I}} = b^{\mathcal{I}} \Leftrightarrow a^{\mathcal{I}_c} = b^{\mathcal{I}_c} \Leftrightarrow \mathcal{I}_c \models a \doteq b \Leftrightarrow \mathcal{I}_c \models \mathcal{S}(a \doteq b)$.
(6) φ is in form of $a \not\doteq b$: $\mathcal{I} \models_s a \not\doteq b \Leftrightarrow a^{\mathcal{I}} \neq b^{\mathcal{I}} \Leftrightarrow a^{\mathcal{I}_c} \neq b^{\mathcal{I}_c} \Leftrightarrow \mathcal{I}_c \models a \not\doteq b \Leftrightarrow \mathcal{I}_c \models \mathcal{S}(a \not\doteq b)$.
(7) φ is in form of $C(a)$: we prove $\mathcal{I} \models_s C(a) \Leftrightarrow \mathcal{I}_c \models \mathcal{S}(C(a))$ by induction on the structure of C.
- (Base step) there exist three cases:
  (i) C is a concept name A: $\mathcal{I} \models_s A(a) \Leftrightarrow a^{\mathcal{I}} \in +A$ where $A^{\mathcal{I}} = \langle +A, -A\rangle \Leftrightarrow a^{\mathcal{I}_c} \in (A^+)^{\mathcal{I}_c} \Leftrightarrow \mathcal{I}_c \models A^+(a) \Leftrightarrow \mathcal{I}_c \models \mathcal{S}(A)(a)$. Then $\mathcal{I} \models_s A(a) \Leftrightarrow \mathcal{I}_c \models \mathcal{S}(A(a))$.
  (ii) C is a datatype d: $\mathcal{I} \models_s d(a) \Leftrightarrow a^{\mathcal{I}} \in +d$ where $d^{\mathcal{I}} = \langle +d, -d\rangle \Leftrightarrow a^{\mathcal{I}_c} \in (d^+)^{\mathcal{I}_c} \Leftrightarrow \mathcal{I}_c \models d^+(a) \Leftrightarrow \mathcal{I}_c \models \mathcal{S}(d)(a,b)$. Then $\mathcal{I} \models_s d(a) \Leftrightarrow \mathcal{I}_c \models \mathcal{S}(d(a))$.
  (iii) C is a nominal $\{o\}$: $\mathcal{I} \models_s \{o\}(a) \Leftrightarrow a^{\mathcal{I}} \in \{o^{\mathcal{I}}\}$, that is, $a^{\mathcal{I}} = o^{\mathcal{I}}$, $\Leftrightarrow a^{\mathcal{I}_c} \in \{o^{\mathcal{I}_c}\} \Leftrightarrow \mathcal{I}_c \models \{o\}(a) \Leftrightarrow \mathcal{I}_c \models \mathcal{S}(\{o\})(a)$. Then $\mathcal{I} \models_s \{o\}(a) \Leftrightarrow \mathcal{I}_c \models \mathcal{S}(\{o\}(a))$.
- (Inductive step) Assume that $\mathcal{I} \models_s C_i(a) \Leftrightarrow \mathcal{I}_c \models \mathcal{S}(C_i(a))$ where $C_i^{\mathcal{I}} = \langle +C_i, -C_i\rangle$ $(i=1,2)$. We discuss the following cases:
  (i) C is in form of $\overline{C_1}$:

$$\begin{aligned}
\mathcal{I} \models_s (\overline{C}_1)(a) \quad &\Leftrightarrow \quad a^{\mathcal{I}} \in +(\overline{C}_1);\\
\Leftrightarrow \quad a^{\mathcal{I}} \in \Delta \setminus +C_1 \quad &\Leftrightarrow \quad a^{\mathcal{I}} \notin +C_1 \quad \Leftrightarrow \quad a^{\mathcal{I}} \notin \mathcal{S}(C_1)^{\mathcal{I}_c};\\
\Leftrightarrow \quad a^{\mathcal{I}} \in \big(\neg\mathcal{S}(C_1)\big)^{\mathcal{I}_c} \quad &\Leftrightarrow \quad a^{\mathcal{I}} \in \big(\mathcal{S}(\overline{C}_1)\big)^{\mathcal{I}_c} \quad \Leftrightarrow \quad \mathcal{I}_c \models \mathcal{S}(\overline{C}_1)(a).
\end{aligned}$$

Then $\mathcal{I} \models_s (\bar{C})(a) \Leftrightarrow \mathcal{I}_c \models \mathcal{S}((\bar{C})(a))$.

(ii) C is in form of $\neg C_1$:

$$\mathcal{I} \models_s (\neg C)(a) \quad \Leftrightarrow \quad a^{\mathcal{I}} \in -C \quad \Leftrightarrow \quad a^{\mathcal{I}_c} \in \left(C^-\right)^{\mathcal{I}_c}; \quad \Leftrightarrow \quad \mathcal{I}_c \models C^-(a) \quad \Leftrightarrow \quad \mathcal{I}_c \models \mathcal{S}(\neg C)(a).$$

Then $\mathcal{I} \models_s (\neg C)(a) \Leftrightarrow \mathcal{I}_c \models \mathcal{S}((\neg C)(a))$.

(iii) C is in form of $C_1 \sqcap C_2$:

$$\begin{aligned}
&\mathcal{I} \models_s C_1 \sqcap C_2(a) \quad \Leftrightarrow \quad a^{\mathcal{I}} \in +(C_1 \sqcap C_2);\\
&\quad \Leftrightarrow \quad a^{\mathcal{I}} \in +C_1 \cap +C_2;\\
&\quad \Leftrightarrow \quad a^{\mathcal{I}_c} \in \mathcal{S}(C_1)^{\mathcal{I}_c} \cap \mathcal{S}(C_2)^{\mathcal{I}_c};\\
&\quad \Leftrightarrow \quad a^{\mathcal{I}_c} \in \mathcal{S}(C_1 \sqcap C_2)^{\mathcal{I}_c};\\
&\quad \Leftrightarrow \quad \mathcal{I}_c \models \mathcal{S}(C_1 \sqcap C_2)(a).
\end{aligned}$$

Then $\mathcal{I} \models_s (C_1 \sqcap C_2)(a) \Leftrightarrow \mathcal{I}_c \models \mathcal{S}((C_1 \sqcap C_2)(a))$.

(iv) C is in form of $C_1 \sqcup C_2$:

$$\begin{aligned}
&\mathcal{I} \models_s (C_1 \sqcup C_2)(a) \quad \Leftrightarrow \quad a^{\mathcal{I}} \in +\left((C_1 \sqcup C_2)^{\mathcal{I}}\right);\\
&\quad \Leftrightarrow \quad a^{\mathcal{I}} \in (+C_1 \cup +C_2) \cap (\overline{-C_1} \cup +C_2) \cap (+C_1 \cup \overline{-C_2});\\
&\quad \Leftrightarrow \quad \left[a^{\mathcal{I}} \in +C_1 \text{ or } a^{\mathcal{I}} \in +C_2)\right] \text{ and } \left[a^{\mathcal{I}} \notin -C_1 \text{ or } a^{\mathcal{I}} \in +C_2)\right] \text{ and } \left[a^{\mathcal{I}} \in +C_1 \text{ or } a \notin -C_2)\right];\\
&\quad \Leftrightarrow \quad \left[a^{\mathcal{I}_c} \in \mathcal{S}(C_1) \text{ or } a^{\mathcal{I}_c} \in \mathcal{S}(C_2))\right] \text{ and } \left[a^{\mathcal{I}} \notin \mathcal{S}(\neg C)^{\mathcal{I}_c} \text{ or } a^{\mathcal{I}} \in \mathcal{S}(C_2)^{\mathcal{I}_c}\right]\\
&\qquad\quad \text{and } \left[a^{\mathcal{I}} \in \mathcal{S}(C_1)^{\mathcal{I}_c} \text{ or } a \notin \mathcal{S}(\neg C_2)^{\mathcal{I}_c})\right];\\
&\quad \Leftrightarrow \quad \left[\mathcal{I}_c \models \left(\mathcal{S}(C_1) \sqcup \mathcal{S}(C_2)\right)(a)\right] \text{ and } \left[\mathcal{I}_c \models \left(\neg\mathcal{S}(\neg C_1) \sqcup \mathcal{S}(C_2)\right)(a)\right]\\
&\qquad\quad \text{and } \left[\mathcal{I}_c \models \left(\mathcal{S}(C_1) \sqcup \neg\mathcal{S}(\neg C_2)\right)(a)\right];\\
&\quad \Leftrightarrow \quad \mathcal{I}_c \models \left(\mathcal{S}(C_1) \sqcup \mathcal{S}(C_2)\right) \sqcap \left(\neg\mathcal{S}(\neg C_1) \sqcup \mathcal{S}(C_2)\right) \sqcap \left(\mathcal{S}(C_1) \sqcup \neg\mathcal{S}(\neg C_2)\right)(a);\\
&\quad \Leftrightarrow \quad \mathcal{I}_c \models \mathcal{S}(C_1 \sqcup C_2)(a).
\end{aligned}$$

Then $\mathcal{I} \models_s (C_1 \sqcup C_2)(a) \Leftrightarrow \mathcal{I}_c \models \mathcal{S}((C_1 \sqcup C_2)(a))$.

(v) C is in form of $\forall R.C_1$.

$$\begin{aligned}
&\mathcal{I} \models_s (\forall R.C_1)(a) \quad \Leftrightarrow \quad \text{for all } b \in \Delta^{\mathcal{I}}, \left(a^{\mathcal{I}}, b\right) \in +R \text{ implies } b \in +C_1;\\
&\quad \Leftrightarrow \quad \text{for all } b \in \Delta^{\mathcal{I}_c}, \left(a^{\mathcal{I}_c}, b\right) \in \mathcal{S}(R)^{\mathcal{I}_c} \text{ implies } b \in \mathcal{S}(C_1)^{\mathcal{I}_c};\\
&\quad \Leftrightarrow \quad \mathcal{I}_c \models \forall\mathcal{S}(R).\mathcal{S}(C_1).
\end{aligned}$$

Then $\mathcal{I} \models_s (\forall R.C_1)(a) \Leftrightarrow \mathcal{I}_c \models \mathcal{S}((\forall R.C_1)(a))$.

Analogously, we can prove $\mathcal{I} \models_s (\forall T.d)(a) \Leftrightarrow \mathcal{I}_c \models \mathcal{S}((\forall T.d)(a))$.

(vi) C is in form of $\exists R.C_1$:

$$\begin{aligned}
&\mathcal{I} \models_s (\exists R.C_1)(a) \quad \Leftrightarrow \quad \text{there exists } b \in \Delta^{\mathcal{I}}, \left(a^{\mathcal{I}}, b\right) \in +R \text{ and } b \in +C_1;\\
&\quad \Leftrightarrow \quad \text{there exists } b \in \Delta^{\mathcal{I}_c}, \left(a^{\mathcal{I}_c}, b\right) \in \mathcal{S}(R)^{\mathcal{I}_c} \text{ and } b \in \mathcal{S}(C_1)^{\mathcal{I}_c};\\
&\quad \Leftrightarrow \quad \mathcal{I}_c \models \exists\mathcal{S}(R).\mathcal{S}(C_1).
\end{aligned}$$

Then $\mathcal{I} \models_s (\exists R.C_1)(a) \Leftrightarrow \mathcal{I}_c \models \mathcal{S}((\exists R.C_1)(a))$.

Analogously, we can prove $\mathcal{I} \models_s (\exists T.d)(a) \Leftrightarrow \mathcal{I}_c \models \mathcal{S}((\exists T.d)(a))$.

(vii) C is in form of $\geqslant n\, S$:

$$\begin{aligned}
&\mathcal{I} \models_s (\geqslant n\, S)(a);\\
&\quad \Leftrightarrow \quad \text{there exists at least } n \text{ instances } b \in \Delta^{\mathcal{I}}, \left(a^{\mathcal{I}}, b\right) \in +S;\\
&\quad \Leftrightarrow \quad \text{there exists at least } n \text{ instances } b \in \Delta^{\mathcal{I}_c}, \left(a^{\mathcal{I}_c}, b\right) \in \mathcal{S}(S)^{\mathcal{I}_c};\\
&\quad \Leftrightarrow \quad \mathcal{I}_c \models \geqslant n\, \mathcal{S}(R).
\end{aligned}$$

Then $\mathcal{I} \models_s (\geqslant n\, S)(a) \Leftrightarrow \mathcal{I}_c \models \mathcal{S}((\geqslant n\, S)(a))$.

Analogously, we can prove $\mathcal{I} \models_s (\leqslant n\, S)(a) \Leftrightarrow \mathcal{I}_c \models \mathcal{S}((\leqslant n\, S)(a))$.

(8) φ is in form of $C_1 \sqsubseteq C_2$. By item (7) of this proof, we can conclude that for any concept C, for any strong interpretation $\mathcal{I}$, $\mathcal{S}(C)^{\mathcal{I}_c} = +C$ and $\mathcal{S}(\neg C)^{\mathcal{I}_c} = -C$ where $C^{\mathcal{I}} = \langle +C, -C\rangle$. That is, for any concept C_i, we have $\mathcal{S}(C_i)^{\mathcal{I}_c} = +C_i$ and $\mathcal{S}(\neg C_i)^{\mathcal{I}_c} = -C_i$ where $C_i^{\mathcal{I}} = \langle +C_i, -C_i\rangle$.

$$\begin{aligned}
\mathcal{I} \models_s C_1 \sqsubseteq C_2 \quad &\Leftrightarrow \quad \overline{-C_1} \subseteq +C_2, +C_1 \subseteq +C_2 \text{ and } -C_2 \subseteq -C_1;\\
&\Leftrightarrow \quad \overline{\big(\mathcal{S}(\neg C_1)\big)^{\mathcal{I}_c}} \subseteq \big(\mathcal{S}(C_2)\big)^{\mathcal{I}_c}, \big(\mathcal{S}(C_1)\big)^{\mathcal{I}_c} \subseteq \big(\mathcal{S}(C_2)\big)^{\mathcal{I}_c} \text{ and } \big(\mathcal{S}(\neg C_2)\big)^{\mathcal{I}_c} \subseteq \big(\mathcal{S}(\neg C_1)\big)^{\mathcal{I}_c};\\
&\Leftrightarrow \quad \big(\neg\mathcal{S}(\neg C_1)\big)^{\mathcal{I}_c} \subseteq \big(\mathcal{S}(C_2)\big)^{\mathcal{I}_c}, \big(\mathcal{S}(C_1)\big)^{\mathcal{I}_c} \subseteq \big(\mathcal{S}(C_2)\big)^{\mathcal{I}_c} \text{ and } \big(\mathcal{S}(\neg C_2)\big)^{\mathcal{I}_c} \subseteq \big(\mathcal{S}(\neg C_1)\big)^{\mathcal{I}_c};\\
&\Leftrightarrow \quad \mathcal{I}_c \models \neg\mathcal{S}(\neg C_1) \sqsubseteq \mathcal{S}(C_2), \mathcal{I}_c \models \mathcal{S}(C_1) \sqsubseteq \mathcal{S}(C_2) \text{ and } \mathcal{I}_c \models \mathcal{S}(\neg C_2) \sqsubseteq \mathcal{S}(\neg C_1).
\end{aligned}$$

Then $\mathcal{I} \models_s C_1 \sqsubseteq C_2 \Leftrightarrow \mathcal{I}_c \models \mathcal{S}(C_1 \sqsubseteq C_2)$.
Analogously, we can prove $\mathcal{I} \models_s C_1 \sqsubseteq C_2 \Leftrightarrow \mathcal{I}_c \models \mathcal{S}(C_1 \sqsubseteq C_2)$ where $X_i \in \{R_i, T_i, d_i\}$ $(i = 1, 2)$.

Therefore, this claim holds, that is, for any φ, $\mathcal{I} \models_s \varphi \Leftrightarrow \mathcal{I}_c \models \mathcal{S}(\varphi)$. That is, for any $\mathcal{K}$, $\mathcal{I} \models_s \mathcal{K} \Leftrightarrow \mathcal{I}_c \models \mathcal{S}(\mathcal{K})$. □

Though our transformation are defined in $\mathcal{SHOIN}(\mathbf{D})$, we can technically extend weak/strong transformation on qualified number restrictions $\mathcal{Q}$ by adding the following rules:

$$\mathcal{X}(\geqslant n\,R.C) = \geqslant n\,\mathcal{X}(R).\mathcal{X}(C); \tag{15}$$

$$\mathcal{X}(\leqslant n\,R.C) = \leqslant n\,\mathcal{X}(R).\mathcal{X}(C); \tag{16}$$

where $\mathcal{X}$ is a placeholder of $\mathcal{W}$ or $\mathcal{S}$.

Indeed, the weak transformation is the four-valued transformation with restricting in internal GCIs proposed by [23,24] to reduce the four-valued DL entailment into the classical DL entailment. Technically, two other four-valued transformations (with restricting in material GCIs and strong GCIs) are also represented by the weak transformation. Formally, the weak transformation for material GCIs (denoted by $\mathcal{W}_m$) and the weak transformation for strong GCIs (denoted by $\mathcal{W}_s$) are the same as the weak transformation $\mathcal{W}$ in all symbols except GCIs as follows: let $X_i \in \{C_i, R_i, T_i, d_i\}$ $(i = 1, 2)$,

$$\mathcal{W}_m(X_1 \mapsto X_2) = \neg\mathcal{W}_m(\neg X_1) \sqsubseteq \mathcal{W}_m(X_2); \tag{17}$$

$$\mathcal{W}_s(X_1 \to X_2) = \big\{\mathcal{W}_s(X_1) \sqsubseteq \mathcal{W}_s(X_2), \neg\mathcal{W}_s(X_2) \sqsubseteq \neg\mathcal{W}_s(X_1)\big\}. \tag{18}$$

As a result, $\mathcal{W}_m$ and $\mathcal{W}_s$ can represent the four-valued transformation with restricting in material GCIs and the four-valued transformation with restricting in strong GCIs respectively.

Indeed, for any four-valued interpretation $\mathcal{I}_4$, if $X_i^{\mathcal{I}_4} = \langle +X_i, -X_i\rangle$ $(i = 1, 2)$ then $\mathcal{I}_4 \models_4 X_1 \mapsto X_2$ if and only if $\overline{-X_1} \subseteq +X_2$ and $\mathcal{I}_4 \models_4 X_1 \to X_2$ if and only if $+X_1 \subseteq +X_2$ and $-X_2 \subseteq -X_1$. For any classical interpretation $\mathcal{I}_c$, we have

(1) $(\neg\mathcal{W}_m(\neg X_1))^{\mathcal{I}_c} = \overline{-X_1}$ and $(\mathcal{W}_m(X_2))^{\mathcal{I}_c} = +X_2$. Then $\mathcal{I}_c \models \neg\mathcal{W}_m(\neg X_1) \sqsubseteq \mathcal{W}_m(X_2)$ since $\overline{-X_1} \subseteq +X_2$.
(2) $(\mathcal{W}_m(\neg X_i))^{\mathcal{I}_c} = -X_i$ and $(\mathcal{W}_m(X_i))^{\mathcal{I}_c} = +X_i$ $(i = 1, 2)$. Then $\mathcal{I}_c \models \mathcal{W}_m(X_1) \sqsubseteq \mathcal{W}_m(X_2)$ and $\mathcal{I}_c \models \mathcal{W}_m(\neg X_2) \sqsubseteq \mathcal{W}_m(\neg X_1)$ since $+X_1 \subseteq +X_2$ and $-X_2 \subseteq -X_1$.

Besides, compared with four-valued transformation, an important contribution of our transformation is the way of dealing with $\sqcup$, which can be seen from the definition of the strong transformation below. Unlike previous transformations, a GCI $C \sqsubseteq D$ (possibly, nominals inclusion or datatype inclusion) is translated into a set of axioms under the strong transformation.

Next, we discuss the complexity of two kinds of transformations from QC-$\mathcal{SHOIN}(\mathbf{D})$ to $\mathcal{SHOIN}(\mathbf{D})$.

It is trivial that weak transformation for concept disjunction/conjunction can be computed in linear time. If we transform QC axioms naively then it might explode in exponential time. By using the following form, we can show that the strong transformation for the concept disjunction $C_1 \sqcup \cdots \sqcup C_n$ can be also computed in linear time.

Proposition 12. *Let $C_1, \ldots, C_n$ be DL concepts. Then*

$$\mathcal{S}(C_1 \sqcup \cdots \sqcup C_n) = \bigsqcup_{i=1}^{n} \big(\mathcal{S}(C_i) \sqcap \neg\mathcal{S}(\neg C_i)\big) \sqcup \bigsqcap_{i=1}^{n} \big(\mathcal{S}(C_i) \sqcap \mathcal{S}(\neg C_i)\big). \tag{19}$$

Proof. We inductively prove this proposition. Let $C_1, \ldots, C_n$ be concepts.

- (Base step) $n = 2$, i.e., the simplest case of $C_1 \sqcup C_2$,

$$\begin{aligned}
&\mathcal{I} \models_s C_1 \sqcup C_2;\\
\Leftrightarrow\ &\big(\mathcal{I} \models_s C_1 \text{ or } \mathcal{I} \models_s C_2\big), \big(\mathcal{I} \not\models_s \neg C_1 \text{ or } \mathcal{I} \models_s C_2\big), (\mathcal{I} \models_s C_1 \text{ or } \mathcal{I} \not\models_s \neg C_2);\\
\Leftrightarrow\ &\big(\mathcal{I}_c \models \mathcal{S}(C_1) \text{ or } \mathcal{I}_c \models_s \mathcal{S}(C_2)\big), \big(\mathcal{I}_c \not\models \mathcal{S}(\neg C_1) \text{ or } \mathcal{I}_c \models \mathcal{S}(C_2)\big), \big(\mathcal{I}_c \models \mathcal{S}(C_1) \text{ or } \mathcal{I}_c \not\models \mathcal{S}(\neg C_2)\big);\\
\Leftrightarrow\ &\big(\mathcal{I}_c \models \mathcal{S}(C_1) \text{ or } \mathcal{I}_c \models_s \mathcal{S}(C_2)\big), \big(\mathcal{I}_c \models \neg\mathcal{S}(\neg C_1) \text{ or } \mathcal{I}_c \models \mathcal{S}(C_2)\big), \big(\mathcal{I}_c \models \mathcal{S}(C_1) \text{ or } \mathcal{I}_c \models \neg\mathcal{S}(\neg C_2)\big);\\
\Leftrightarrow\ &\mathcal{I}_c \models \big(\big(\mathcal{S}(C_1) \sqcap \neg\mathcal{S}(\neg C_1)\big) \sqcup \big(\mathcal{S}(C_2) \sqcap \neg\mathcal{S}(\neg C_2)\big)\big) \sqcup \big(\mathcal{S}(C_1) \sqcap \mathcal{S}(\neg C_1) \sqcap \mathcal{S}(C_2) \sqcap \mathcal{S}(\neg C_2)\big).
\end{aligned}$$

- (Inductive step) Assume that Eq. (19) holds for n concepts:

$$\mathcal{S}(C_1 \sqcup \cdots \sqcup C_n) = \bigsqcup_{i=1}^{n}\big(\mathcal{S}(C_i) \sqcap \neg\mathcal{S}(\neg C_i)\big) \sqcup \bigsqcap_{i=1}^{n}\big(\mathcal{S}(C_i) \sqcap \mathcal{S}(\neg C_i)\big). \tag{20}$$

Next, we consider $\mathcal{S}(C_1 \sqcup \cdots \sqcup C_n \sqcup C_{n+1})$. Let $C = C_1 \sqcup \cdots \sqcup C_n$ and $C_1 \sqcup \cdots \sqcup C_n \sqcup C_{n+1} = C \sqcup C_{n+1}$. We have $\mathcal{S}(C_1 \sqcap \cdots \sqcap C_n) = \mathcal{S}(C_1) \sqcap \cdots \sqcap \mathcal{S}(C_n)$.

$$\begin{aligned}
&\mathcal{I} \models_s C \sqcup C_{n+1} \quad \text{(by using the analogous proof in the base step)};\\
\Leftrightarrow\ &\mathcal{I}_c \models \big(\big(\mathcal{S}(C) \sqcap \neg\mathcal{S}(\neg C)\big) \sqcup \big(\mathcal{S}(C_{n+1}) \sqcap \neg\mathcal{S}(\neg C_{n+1})\big)\big) \sqcup \big(\mathcal{S}(C) \sqcap \mathcal{S}(\neg C) \sqcap \mathcal{S}(C_{n+1}) \sqcap \mathcal{S}(\neg C_{n+1})\big);\\
\Leftrightarrow\ &\mathcal{I}_c \models \Big(\Big(\bigsqcup_{i=1}^{n}\big(\mathcal{S}(C_i) \sqcap \neg\mathcal{S}(\neg C_i)\big) \sqcup \bigsqcap_{i=1}^{n}\big(\mathcal{S}(C_i) \sqcap \mathcal{S}(\neg C_i)\big)\Big) \sqcap\\
&\neg\mathcal{S}(\neg C_1 \sqcap \cdots \sqcap \neg C_n)\Big) \sqcup \big(\mathcal{S}(C_{n+1}) \sqcap \neg\mathcal{S}(\neg C_{n+1})\big)\big) \sqcup\\
&\Big(\Big(\bigsqcup_{i=1}^{n}\big(\mathcal{S}(C_i) \sqcap \neg\mathcal{S}(\neg C_i)\big) \sqcup \bigsqcap_{i=1}^{n}\big(\mathcal{S}(C_i) \sqcap \mathcal{S}(\neg C_i)\big)\Big) \sqcap\\
&\mathcal{S}(\neg C_1 \sqcap \cdots \sqcap \neg C_n)\Big) \sqcap \big(\mathcal{S}(C_{n+1}) \sqcap \mathcal{S}(\neg C_{n+1})\big));\\
\Leftrightarrow\ &\mathcal{I}_c \models \Big(\Big(\bigsqcup_{i=1}^{n}\big(\mathcal{S}(C_i) \sqcap \neg\mathcal{S}(\neg C_i)\big) \sqcap \big(\neg\mathcal{S}(\neg C_1) \sqcup \cdots \sqcup \neg\mathcal{S}(\neg C_n)\big)\Big) \sqcup\\
&\Big(\bigsqcap_{i=1}^{n}\big(\mathcal{S}(C_i) \sqcap \mathcal{S}(\neg C_i)\big) \sqcap \big(\neg\mathcal{S}(\neg C_1) \sqcup \cdots \sqcup \neg\mathcal{S}(\neg C_n)\big)\Big) \sqcup \big(\mathcal{S}(C_{n+1}) \sqcap \neg\mathcal{S}(\neg C_{n+1})\big)\Big) \sqcup\\
&\Big(\Big(\bigsqcup_{i=1}^{n}\big(\mathcal{S}(C_i) \sqcap \neg\mathcal{S}(\neg C_i)\big) \sqcap \big(\mathcal{S}(\neg C_1) \sqcap \cdots \sqcap \mathcal{S}(\neg C_n)\big)\Big) \sqcup\\
&\Big(\Big(\bigsqcap_{i=1}^{n}\big(\mathcal{S}(C_i) \sqcap \mathcal{S}(\neg C_i)\big)\Big) \sqcap \big(\mathcal{S}(\neg C_1) \sqcap \cdots \sqcap \mathcal{S}(\neg C_n)\big)\Big) \sqcap \big(\mathcal{S}(C_{n+1}) \sqcap \mathcal{S}(\neg C_{n+1})\big)\Big);\\
\Leftrightarrow\ &\mathcal{I}_c \models (\Big(\bigsqcup_{i=1}^{n}\big(\mathcal{S}(C_i) \sqcup \neg\mathcal{S}(\neg C_i)\big) \sqcup \big(\mathcal{S}(C_{n+1}) \sqcap \neg\mathcal{S}(\neg C_{n+1})\big)\Big) \sqcup\\
&\Big(\bigsqcap_{i=1}^{n}\big(\mathcal{S}(C_i) \sqcap \mathcal{S}(\neg C_i)\big)\Big) \sqcap \big(\mathcal{S}(C_{n+1}) \sqcap \mathcal{S}(\neg C_{n+1})\big);\\
\Leftrightarrow\ &\mathcal{I}_c \models \bigsqcup_{i=1}^{n+1}\big(\mathcal{S}(C_i) \sqcap \neg\mathcal{S}(\neg C_i)\big) \sqcup \bigsqcap_{i=1}^{n+1}\big(\mathcal{S}(C_i) \sqcap \mathcal{S}(\neg C_i)\big).
\end{aligned}$$

Then, Eq. (19) holds for $n+1$ concepts. Therefore, Eq. (19) holds for disjunction of arbitrary many concepts. □

By applying two transformations, the language of QC-$\mathcal{SHOIN}(\mathbf{D})$ is translated into the language of $\mathcal{SHOIN}(\mathbf{D})$ in a polynomial time.

Next, we apply two transformations (strong transformation and weak transformation) to transform the QC consistency problem and the QC entailment problem into the (classical) consistency problem and the entailment problem respectively.

Theorem 1. *Let $\mathcal{K}$ be a QC KB and φ a QC axiom. Then*

(1) *$\mathcal{K}$ is QC consistent if and only if $\mathcal{S}(\mathcal{K})$ is consistent*;
(2) *$\mathcal{K} \models_Q \varphi$ if and only if $\mathcal{S}(\mathcal{K}) \models \mathcal{W}(\varphi)$.*

Proof. In [25], the QC entailment can be equivalently reduced into classical entailment in propositional logic. Next, we apply this technique to prove this theorem.

By the proof of Proposition 11, every base interpretation $\mathcal{I}$ on language $\mathcal{L}$ can also viewed as a classical interpretation $\mathcal{I}_c$ which is on language $\mathcal{L}^{\pm}$ such that $\mathcal{I}_c \models C(a)$ if and only if $a^{\mathcal{I}} \in +C$ where $C^{\mathcal{I}} = \langle +C, -C\rangle$. [24] shows that $\mathcal{I}$ satisfies axioms under four-valued semantics if and only if $\mathcal{I}_c$ satisfies axioms under classical semantics.

By Proposition 11, the first item of this proposition is trivial. Next, we prove the second item. $\mathcal{I} \models_s \mathcal{K} \Leftrightarrow \mathcal{I}_c \models \mathcal{S}(\mathcal{K})$ and $\mathcal{I} \models_w \varphi \Leftrightarrow \mathcal{I}_c \models \mathcal{W}(\varphi)$. So

$$\begin{aligned}
&\mathcal{K} \models_Q \varphi;\\
\Leftrightarrow\quad &\text{for all } \mathcal{I}, \text{if } \mathcal{I} \models_s \mathcal{K} \text{ then } \mathcal{I} \models_w \varphi;\\
\Leftrightarrow\quad &\text{for all } \mathcal{I}_c, \text{if } \mathcal{I}_c \models \mathcal{S}(\mathcal{K}) \text{ then } \mathcal{I}_c \models \mathcal{W}(\varphi);\\
\Leftrightarrow\quad &\mathcal{S}(\mathcal{K}) \vdash \mathcal{W}(\varphi). \qquad \square
\end{aligned}$$

Two transformations can accurately preserve the QC semantics during translating QCDLs into DLs.

Because two kinds of QC entailment problems can be reduced into the QC consistency problem (see Proposition 10), the QC entailment problem can be reduced into the consistency problem.

Corollary 1. *Let $\mathcal{T}$ be a terminology, $\mathcal{R}$ a role hierarchy, $\mathcal{A}$ an ABox and C, D concepts. Then*

(1) *$(\mathcal{T}, \mathcal{R}, \mathcal{A}\}) \models_Q C(a)$ is QC inconsistent w.r.t. $\mathcal{R}_U$ if and only if $(\mathcal{S}(\mathcal{T}), \mathcal{S}(\mathcal{R}), \mathcal{S}(\mathcal{A}) \cup \{\neg\mathcal{W}(C)(a)\})$ is inconsistent w.r.t. $\mathcal{R}_U$;*
(2) *$(\mathcal{T}, \mathcal{R}, \emptyset) \models_Q C \sqsubseteq D$ is inconsistent w.r.t. $\mathcal{R}_U$ if and only if $(\mathcal{S}(\mathcal{T}), \mathcal{S}(\mathcal{R}), \{(\mathcal{W}(C) \sqcap \neg\mathcal{W}(D))(\iota)\})$ is QC inconsistent w.r.t. $\mathcal{R}_U$ for some new individual $\iota \in \Delta$.*

Proof. It follows from Proposition 11, Proposition 10 and Theorem 1. $\square$

Corollary 1 provides the theoretical support of adopting the off-the-shelf DL OWL reasoners to determine whether $\mathcal{K} \models_Q \varphi$ (an implement system based on this theory is presented in Section 5).

A *transformation-based algorithm* for the QC entailment starts a KB $\mathcal{K}$ and a query φ and then applies the strong transformation $\mathcal{S}(\cdot)$ on $\mathcal{K}$ and the weak transformation $\mathcal{W}(\cdot)$ on φ and calls some off-the-shelf OWL reasoner to checking whether $\mathcal{S}(\mathcal{K}) \models \mathcal{W}(\varphi)$.

For instance, let $\mathcal{K} = (\{Bird \sqsubseteq Fly, Penguin \sqsubseteq Bird, Penguin \sqsubseteq \neg Fly, Swallow \sqsubseteq Bird\}, \{Penguin(tweety), Swallow(fred)\})$ be a KB, called *BirdKB*.

(1) $\mathcal{K}$ will be transformed to

$$\begin{aligned}
\mathcal{S}(\mathcal{K}) = \big(\big\{&Bird^+ \sqsubseteq Fly^+, Fly^- \sqsubseteq Bird^-, \neg Bird^- \sqsubseteq Fly^+,\\
&Penguin^+ \sqsubseteq Bird^+, Bird^- \sqsubseteq Penguin^-, \neg Penguin^- \sqsubseteq Bird^+,\\
&Penguin^+ \sqsubseteq -Fly, Fly^+ \sqsubseteq Penguin^-, \neg Penguin^- \sqsubseteq Fly^-,\\
&Swallow^+ \sqsubseteq Bird^+, Bird^- \sqsubseteq Swallow^-, \neg Swallow^- \sqsubseteq Bird^+\big\},\\
&\big\{Penguin^+(tweety), Swallow^+(fred)\big\}\big).
\end{aligned}$$

(2) The query $\varphi = Fly(fred)$ will be transformed to $\mathcal{W}(\varphi) = Fly(fred)^+$.
(3) Querying $\mathcal{K} \models_Q Fly(fred)$ can be reduced to $\mathcal{S}(\mathcal{K}) \models Fly^+(fred)$. As a result, the answer to *Fly(fred)* is "yes" since $\mathcal{S}(\mathcal{K}) \cup \{\neg Fly^+(fred)\}$ is inconsistent, that is, $\mathcal{S}(\mathcal{K}) \models Fly^+(fred)$ by Lemma 1. These knowledge (*Swallow(fred)* and $Swallow \sqsubseteq Bird$ and $Bird \sqsubseteq Fly$) representing *fred* does not contain conflict although those knowledge (*Penguin(tweety)*, $Penguin \sqsubseteq Bird$, $Bird \sqsubseteq Fly$ and $Penguin \sqsubseteq \neg Fly$) representing *tweety* contains some conflict. Intuitively, it is reasonable that *fred* can fly.

In the rest of this section, we discuss the computational complexity of checking problems in QC-$\mathcal{SHOIN}(\mathbf{D})$. Because the strong and weak transformations can be obtained in linear time and the scope of the new KB $\mathcal{S}(\mathcal{K})$ will be polynomially increased, the complexity of the QC entailment problem is not higher than that of the classical entailment problem in $\mathcal{SHOIN}(\mathbf{D})$ which is NEXPTIME-Complete [1].

Theorem 2. *Let $\mathcal{K}$ be a QC KB. The problem of determining whether $\mathcal{K}$ is QC consistent is NEXPTIME-Complete.*

Proof.

(1) We show that the QC consistency problem of $\mathcal{K}$ is in NEXPTIME. To do so, we need to show that the QC consistency problem can be reduced into the consistency problem.
   (a) It follows from Theorem 1, $\mathcal{K}$ is QC consistent if and only if $\mathcal{S}(\mathcal{K})$ is consistent.
   (b) In our transformation, connectives $\sqcup$ and $\sqcap$ (here $\sqcap$ occurs together with $\neg$) will cause new concepts with more $\sqcup$ and $\sqcap$ while the $\sqsubseteq$ connective will bring more new axioms. The $\sqsubseteq$ connective can be expressed by $\sqcup$ and $\sqcap$ in Proposition 4. Proposition 12 shows that the scope of $\mathcal{S}(\mathcal{K})$ will in the polynomial increase compared to the scope of $\mathcal{K}$. Therefore, the QC consistency problem of $\mathcal{K}$ is in NEXPTIME since the consistency problem of an $\mathcal{SHOIN}(\mathbf{D})$ KB is in NEXPTIME.

(2) We will claim that the QC consistency problem of $\mathcal{K}$ is NEXPTIME-hard by reducing the consistency problem into the QC consistency problem.
Formally, for any KB $\mathcal{K}$ in $\mathcal{SHOIN}(\mathbf{D})$, $\mathcal{K}$ is classical consistent if and only if $\lambda(\mathcal{K})$ is QC consistent where $\lambda(\mathcal{K})$ is obtained by replacing all $\neg X$ occurring in $\mathcal{K}$ with $\overline{X}$ where $X \in \{C, d, \{o\}\}$. Note that λ can be taken as a transformation and we also introduce λ^{-1} denotes the inverse of λ. Obviously, $\lambda(\mathcal{K})$ is a QC KB in QC-$\mathcal{SHOIN}(\mathbf{D})$ without $\neg$.
(1) if $\lambda(\mathcal{K})$ is QC consistent then there exists some strong model $\mathcal{I}_s$ of $\lambda(\mathcal{K})$, i.e., $\mathcal{I}_s \models \lambda(\phi)$ for all $\lambda(\varphi) \in \lambda(\mathcal{K})$.
Let $\mathcal{I}_s^2$ is obtained from $\mathcal{I}_s$ as follows:

- $\Delta^{\mathcal{I}_s^2} = \Delta^{\mathcal{I}_s}$;
- $\Delta_{\mathbf{D}}^{\mathcal{I}_s^2} = \Delta_{\mathbf{D}}^{\mathcal{I}_s}$;
- $\top^{\mathcal{I}_s^2} = \Delta^{\mathcal{I}_s^2}$;
- $\bot^{\mathcal{I}_s^2} = \emptyset$;
- $a^{\mathcal{I}_s^2} = a^{\mathcal{I}_s}$ for any a;
- $X^{\mathcal{I}_s^2} = +X^{\mathcal{I}_s}$ where $X^{\mathcal{I}_s} = \langle +X, -X\rangle$ and $X \in \{A, R, T, d\}$;
- $\{o\}^{\mathcal{I}_s^2} = \{o^{\mathcal{I}_s}\}$;
- complex concepts are defined following Table 1.

We need to prove that $\mathcal{I}_s^2$ is a model of $\mathcal{K}$, i.e., for all $\varphi \in \mathcal{K}$, $\mathcal{I}_s^2 \models \varphi$.
When φ is a transitive axiom ($Trans(R)$), equality assertion ($a \doteq b$), inequality assertion ($a \not\doteq b$), or role assertion $R(a, b)$, $T(a, d)$, this claim is true. When φ is a concept assertion $C(a)$ and concept inclusion $C_1 \sqsubseteq C_2$,
(a) If $\mathcal{I}_s \models_s (A_1 \sqcup A_2)(a)$, i.e., $a \in +(A_1 \sqcup A_2)^{\mathcal{I}_s}$, $a \in +(\overline{\neg A_1} \sqcup A_2)^{\mathcal{I}_s}$ and $a \in +(A_1 \sqcup \overline{\neg A_2})^{\mathcal{I}_s}$ since $+\overline{\neg A_i}^{\mathcal{I}_s} = (A_i)^{\mathcal{I}_s^2}$ then $a \in (A_1 \sqcup A_2)^{\mathcal{I}_s^2}$, that is, $\mathcal{I}_s^2 \models (A_1 \sqcup A_2)(a)$. We can inductively show that if $\mathcal{I}_s \models_s (C_1 \sqcup C_2)(a)$ then $\mathcal{I}_s^2 \models_s (\lambda^{-1}(C_1) \sqcup \lambda^{-1}(C_2))(a)$ since $+\overline{C}^{\mathcal{I}_s} = (\neg C)^{\mathcal{I}_s^2}$.
(b) If $\mathcal{I}_s \models_s A_1 \sqsubseteq A_2$, that is, $\Delta^{\mathcal{I}_s} \setminus -A_1^{\mathcal{I}_s} \subseteq +A_2^{\mathcal{I}_s}$, $+A_1^{\mathcal{I}_s} \subseteq +A_2^{\mathcal{I}_s}$ and $-A_2^{\mathcal{I}_s} \subseteq -A_2^{\mathcal{I}_s}$ since $+\overline{\neg A_i}^{\mathcal{I}_s} = (A_i)^{\mathcal{I}_s^2}$ then $A_1^{\mathcal{I}_s^2} \subseteq A^{\mathcal{I}_s^2}$, that is, $\mathcal{I}_s^2 \models A_1 \sqsubseteq A_2$. We can inductively show that if $\mathcal{I}_s \models \lambda(C_1) \sqsubseteq \lambda^{-1}(C_2)$ since $+\overline{C_i}^{\mathcal{I}_s} = (\neg C_i)^{\mathcal{I}_s^2}$ $(i = 1, 2)$. Thus $\mathcal{I}_s^2 \models_s C_1 \sqsubseteq C_2$.

Analogously, we can prove that φ is in forms of $\neg C(a)$, $\neg d(a)$, $(C_1 \sqcap C_2)(a)$, $\eta\, R.C(a)$, $\eta\, T.d(a)$ $(\eta \in \{\forall, \exists\})$ and $d_1 \sqsubseteq d_2$. Therefore, $\mathcal{I}_s^2$ is a strong model of $\lambda(\mathcal{K})$ by definition, that is, $\lambda(\mathcal{K})$ is QC consistent.
(2) If $\mathcal{K}$ is classical consistent then there exists some classical model $\mathcal{I}_c$ of $\mathcal{K}$, i.e., $\mathcal{I}_c \models \varphi$ for all $\varphi \in \mathcal{K}$. Note that $\lambda(\varphi)$ mapping to φ is in $\lambda(\mathcal{K})$.
Let $\mathcal{I}_c^Q$ be a strong interpretation (called the *induced BI* of $\mathcal{I}_c$) which is obtained from $\mathcal{I}_c$ as follows:

- $\Delta^{\mathcal{I}_c^Q} = \Delta^{\mathcal{I}_c}$;
- $\Delta_{\mathbf{D}}^{\mathcal{I}_c^Q} = \Delta_{\mathbf{D}}^{\mathcal{I}_c}$;
- $a^{\mathcal{I}_c^Q} = a^{\mathcal{I}_c}$ for any a;
- $\top^{\mathcal{I}_c^Q} = \langle \Delta^{\mathcal{I}_c}, \emptyset\rangle$;
- $\bot^{\mathcal{I}_c^Q} = \langle \emptyset, \Delta^{\mathcal{I}_c}\rangle$;
- $A^{\mathcal{I}_c^Q} = \langle A^{\mathcal{I}_c}, \neg A^{\mathcal{I}_c}\rangle$;
- $R^{\mathcal{I}_c^Q} = \langle R^{\mathcal{I}_c}, (\Delta^{\mathcal{I}_c} \times \Delta^{\mathcal{I}_c}) \setminus R^{\mathcal{I}_c}\rangle$;
- $T^{\mathcal{I}_c^Q} = \langle T^{\mathcal{I}_c}, (\Delta^{\mathcal{I}_c} \times \Delta_{\mathbf{D}}^{\mathcal{I}_c}) \setminus T^{\mathcal{I}_c}\rangle$;
- $\{o\}^{\mathcal{I}_c^Q} = \langle \{o^{\mathcal{I}_c}\}, N\rangle$ where $N \subseteq \Delta^{\mathcal{I}_c}$;
- $d^{\mathbf{D}} = \langle d^{\mathbf{D}}, \Delta_{\mathbf{D}}^{\mathcal{I}_c} \setminus d^{\mathbf{D}}\rangle$;
- complex concepts are defined following Definition 4.

We need to prove that $\mathcal{I}_c^Q$ is a strong model of $\lambda(\mathcal{K})$, i.e., for all $\varphi \in \mathcal{K}$, $\mathcal{I}_c^Q \models_s \lambda(\varphi)$.
When φ is a transitive axiom ($Trans(R)$), equality assertion ($a \doteq b$), inequality assertion ($a \not\doteq b$), or role assertion $R(a, b)$, $T(a, d)$, this claim is true. When φ is a concept assertion $C(a)$ and concept inclusion $C_1 \sqsubseteq C_2$,
(a) $\mathcal{I}_c \models (C_1 \sqcup C_2)(a)$, i.e., $a \in (C_1 \sqcup C_2)^{\mathcal{I}_c}$ then $a \in +(\lambda(C_1))^{\mathcal{I}_c}$ or $a \in +(\lambda(C_2))^{\mathcal{I}_c}$ then $\mathcal{I}_c^Q \models_s \lambda(C_1 \sqcup C_2)(a)$ since $C_i^{\mathcal{I}_c}$ and $\neg C_i^{\mathcal{I}_c}$ are complementary by Definition 4, $C_i^{\mathcal{I}_c} = +(\lambda(C_i))^{\mathcal{I}_c^Q}$ and $\neg C_i^{\mathcal{I}_c} = -(\lambda(C_i))^{\mathcal{I}_c^Q}$ $(i = 1, 2)$.
(b) $\mathcal{I}_c \models C_1 \sqsubseteq C_2$, that is, $C_1^{\mathcal{I}_c} \subseteq C_2^{\mathcal{I}_c}$ then $\Delta^{\mathcal{I}_c^Q} \setminus -(\lambda(C_1))^{\mathcal{I}_c^Q} \subseteq +(\lambda(C_2))^{\mathcal{I}_c^Q}$, $+(\lambda(C_1))^{\mathcal{I}_c^Q} \subseteq +(\lambda(C_2))^{\mathcal{I}_c^Q}$ and $-(\lambda(C_2))^{\mathcal{I}_c^Q} \subseteq -(\lambda(C_1))^{\mathcal{I}_c^Q}$ since $C_i^{\mathcal{I}_c}$ and $\neg C_i^{\mathcal{I}_c}$ are complementary by Definition 4, $C_i^{\mathcal{I}_c} = +(\lambda(C_i))^{\mathcal{I}_c^Q}$ and $\neg C_i^{\mathcal{I}_c} = -(\lambda(C_i))^{\mathcal{I}_c^Q}$ $(i = 1, 2)$. Thus $\mathcal{I}_c^Q \models_s \lambda(C_1 \sqsubseteq C_2)$.

Analogously, we can prove that φ is in forms of $\overline{C}(a)$, $\overline{d}(a)$, $(C_1 \sqcap C_2)(a)$, $\eta R.C(a)$, $\eta T.d(a)$ $(\eta \in \{\forall, \exists\})$ and $d_1 \sqsubseteq d_2$. Therefore, $\mathcal{I}_c^Q$ is a strong model of $\lambda(\mathcal{K})$ by definition, that is, $\lambda(\mathcal{K})$ is QC consistent.

We conclude that the QC entailment problem is NEXPTIME-Complete since the classical entailment is NEXPTIME-Complete in $\mathcal{SHOIN}(\mathbf{D})$ [1]. $\square$

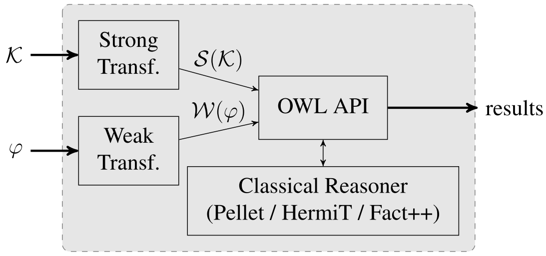

Fig. 1. PROSE architecture.

In addition, because the QC entailment problem can be reduced into the QC consistency problem by Proposition 10 and item (2) of Theorem 1, the QC entailment problem is no higher than the classical entailment problem in DLs.

5. Experiments

5.1. Prototype system

Based on the transformation-based algorithm introduced in the previous section, we have implemented a prototype system for QC-$\mathcal{SHOIN}$(**D**), called PROSE (paraconsistent reasoning on semantic web. The architecture of PROSE is shown in Fig. 1. PROSE is designed in the Decorator Pattern extending the inner classical OWL reasoner so that paraconsistent reasoning on inconsistent KBs can be performed. The module *Strong Transformer* rewrites the input KB $\mathcal{K}$ to a new KB $\mathcal{S}(\mathcal{K})$ while the module *Weak Transformer* changes the input query φ into a new query $\mathcal{W}(\varphi)$. Then the inner classical OWL reasoner is called.

PROSE targets on OWL-API 3.2.4 [14], which is a Java interface and implementation for OWL. OWL-API is supported by many OWL DL Reasoner such as Pellet [37], FaCT++ [42] and HermiT [36]. In the current version of PROSE, the inner classical DL reasoner is Pellet, which is a widely used open source reasoner for OWL 2 DL. However, PROSE can be easily adapted to other DL reasoners. An online demo for PROSE is developed based on GWT and it is available at website: http://prose-web.appspot.com/.

The design of PROSE is inspired by ParOWL [31,23] but with several advantages over ParOWL:

(1) PROSE is flexible: instead of using the strong transformation on the KB and the weak transformation on the query, thus obtaining QC semantics, we can also apply the weak transformation $\mathcal{W}$ including $\mathcal{W}_m$ and $\mathcal{W}_s$ to the KB, obtaining the weak or four-valued semantics for three kinds of GCIs, or apply strong transformation to the query, obtaining the strong semantics;
(2) As we have explained before, PROSE is more powerful in terms of reasoning abilities, even for classical consistent KBs;
(3) From the API point of view, ParOWL only uses closed source KAON2 [27] as underlying reasoner while PROSE targets on OWL-API which is open sourced and supported by many modern OWL Reasoners such as Pellet, Fact++ and HermiT.

5.2. Evaluation and results analysis

5.2.1. Evaluation

The experiments were performed on a Notebook with Intel T2400 1.83G CPU and 2G memory running on Windows 7. The program were written in Java 1.6 with 512M memory allocated for JVM.

We used some benchmarks (including consistent and inconsistent KBs) from Pellet and TONES Ontology Repository [41]. Preliminary experiment results of some benchmark KBs can be found shown in Table 4 where we use $\sharp(N_C)$ to denote the number of concepts, $\sharp(N_R)$ to denote the number of roles and $\sharp(N_I)$ to denote the number of individual names occurring in KBs. The column of **Con** is for the consistency, and the column of **Tran** is the time (in seconds) of the strong transformation.

For the first three consistent KBs, PROSE returns the same reasoning results as Pellet. However, the other KBs in the table are classical inconsistent and thus Pellet is unable to perform any reasoning on them.

5.2.2. Results analysis

We will analyze the results of PROSE by comparing with those of ParOWL in a unified way. To do so, we slightly abuse the truth values in Belnap's bilattice [2] so that our results look more intuitive.

Let $\mathcal{K}$ be a KB. For any query $C(a)$, there exist four possible results, namely "B" (both), "U" (unknown), "T" (true) and "F" (false), of querying over $\mathcal{K}$ as follows:

- B denotes $\mathcal{K} \models_Q C(a)$ and $\mathcal{K} \models_Q \neg C(a)$;
- T denotes $\mathcal{K} \models_Q C(a)$ and $\mathcal{K} \not\models_Q \neg C(a)$;

Table 4
Evaluation on Benchmark KBs.

KB name	DL	$\sharp(N_C)$	$\sharp(N_R)$	$\sharp(N_I)$	Con	Tran (s)
mindswappers	$\mathcal{ALCIF}(\mathbf{D})$	48	60	122	yes	0.468
financial	$\mathcal{ALCOF}$	60	16	17941	yes	3.282
pizza	$\mathcal{SHOIN}$	98	8	5	yes	2.184
particle	$\mathcal{ALCQ}$	73	5	0	yes	2.196
OBO_REL	$\mathcal{SHOIF}$	1110	13	999	yes	12.112
fly_anatomy	$\mathcal{EL}$++	6222	2	0	yes	12.558
Bird	$\mathcal{ALC}$	5	0	2	no	0.015
buggyPolicy	$\mathcal{ALCHO}$	15	3	1	no	0.014
bad − food	$\mathcal{ALCO}(\mathbf{D})$	18	2	2	no	0.02
CHEM	$\mathcal{ALCHOF}(\mathbf{D})$	48	20	1	no	0.069

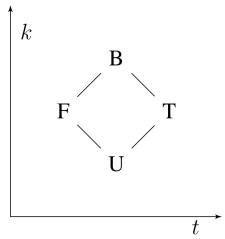

Fig. 2. Belnap's bilattice.

Table 5
Queries and results of School KB.

Query	ParOWL			PROSE
	$\mapsto$	$\sqsubset$	$\rightarrow$	
PhDStudent(*Wade*)	T	T	B	B
Student(*Wade*)	U	T	B	B
Staff(*Wade*)	U	B	B	B
Professor(*Wade*)	U	U	U	F
PhDStudent(*Jack*)	U	U	F	F
Student(*Jack*)	U	U	U	T
Staff(*Jack*)	F	F	F	F
Professor(*Jack*)	U	U	U	F

- F denotes $\mathcal{K} \not\models_Q C(a)$ and $\mathcal{K} \models_Q \neg C(a)$;
- U denotes $\mathcal{K} \not\models_Q C(a)$ and $\mathcal{K} \not\models_Q \neg C(a)$.

For any query $C \sqsubseteq D$, we can also introduce four possible results as follows:

- B denotes $\mathcal{K} \models_Q C \sqsubseteq D$ and $\mathcal{K} \models_Q C \sqsubseteq \neg D$;
- T denotes $\mathcal{K} \models_Q C \sqsubseteq D$ and $\mathcal{K} \not\models_Q C \sqsubseteq \neg D$;
- F denotes $\mathcal{K} \not\models_Q C \sqsubseteq D$ and $\mathcal{K} \models_Q C \sqsubseteq \neg D$;
- U denotes $\mathcal{K} \not\models_Q C \sqsubseteq D$ and $\mathcal{K} \not\models_Q C \sqsubseteq \neg D$.

The result "B" of $C(a)$ indicates the inconsistency caused by $\{C(a), \neg C(a)\}$ while the result "B" of $C \sqsubseteq D$ indicates the incoherency caused by $\{C \sqsubseteq D, C \sqsubseteq \neg D\}$ where for any concept name A, if $A \sqsubseteq C$ then A is unsatisfiable.

The four results can be represented in Belnap's bilattice shown in Fig. 2 where "T" and "F" are two extreme values in the $\preccurlyeq_t$ (the truth ordering) and "B" and "U" are taken as two extreme values in the $\preccurlyeq_k$ (knowledge ordering).

Now, we compare querying results of PROSE with ParOWL by the following two KBs.

1. **School** KB. Let $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ be a KB where $\mathcal{T} = \{PhDStudent \sqsubseteq Student, PhDStudent \sqsubseteq Staff, Student \sqsubseteq \neg Staff, Member \sqsubseteq Student \sqcup Staff, Professor \sqsubseteq Staff\}$ and $\mathcal{A} = \{Member(Wade), Member(Jack), PhDStudent(Wade), \neg Staff(Jack)\}$. Intuitively, $\mathcal{K}$ states that all PhD students are students, all PhD students are also staffs, students and staffs are disjoint, a member is either a student or a staff, and all professor are staffs. Additionally for members, *Wade* is a PhD student and *Jack* is not a staff.

The inconsistency of School KB is caused by both unsatisfiability of *PhDStudent* and *Wade* is an instance of *PhDStudent*.

The results of eight queries in PROSE and ParOWL for three kinds of GCIs are shown in Table 5.

Table 6
Queries and results of Pizza KB.

Query	ParOWL			PROSE
	$\mapsto$	$\sqsubset$	$\rightarrow$	
$IceCream \sqsubseteq Pizza$	U	U	U	B
$\exists hasTopping \sqsubseteq Pizza$	U	U	U	T
$IceCream \sqsubseteq \exists hasTopping$	U	T	T	T
$CheeseyVegetableTopping \sqsubseteq American$	U	T	T	F
$CheeseyVegetableTopping \sqsubseteq Pizza$	U	T	T	F
$American \sqsubseteq Pizza$	U	T	T	T
$Soho \sqsubseteq Pizza$	U	T	T	T
$AmericanHot \sqsubseteq SpicyPizza$	U	T	T	T

2. **Pizza** KB[2] is incoherent but consistent. That is, Pizza KB contains two unsatisfiable concept names, namely, *IceCream* and *CheeseyVegetableTopping*, which are caused by the modeling errors. The first incoherency of Pizza KB is to demonstrate mistakes made with setting a role domain. *hasTopping* has a domain of *Pizza*. This means that the reasoner can infer that all individuals using the *hasTopping* role must be of type *Pizza*. Because of this restriction, all members of *IceCream* must use the *hasTopping* role, and therefore must also be members of *Pizza*. However, *Pizza* and *IceCream* are disjoint, so this causes the unsatisfiability of *Pizza*. Analogously, *CheeseyVegetableTopping* is unsatisfiable since it is a subconcept of both *CheeseyTopping* and *VegetableTopping* but these two concepts are disjoint.

In this experiment, we are interested in TBox reasoning. The results of PROSE and ParOWL for three kinds of GCIs are shown in Table 6.

From Table 5 and Table 6, we have the following observations:

(1) PROSE infers more knowledge following from the knowledge order in Belnap's Bilattice (Fig. 2).
- In School KB, the results of PROSE contain no "U" while those of ParOWL contain at least one "U".
- In Pizza KB, the results of PROSE contain no "U" while those of ParOWL contain at least two "U".

(2) PROSE draws more conclusions with classical truth value.
- In School KB, the results of PROSE contain five "T" or "F" while those of ParOWL contain at most three "T" or "F".
- In Pizza KB, the results of PROSE contain seven "T" or "F" while those of ParOWL contain at most six "T" or "F".

(3) PROSE handles inconsistency or incoherency in a more rational way.
- In School KB, the knowledge about *Jack* is consistent while the knowledge about *Wade* contains some inconsistencies. Accordingly, we can obtain exact conclusions about both of them in PROSE and ParOWL with strong GCIs. Moreover, we can infer that *Jack* is a student since we know *Jack* is not a staff and a member is either a student or a staff while we cannot draw such a conclusion in ParOWL no matter which kinds of GCIs will be selected.
- In Pizza KB, for simplicity, we abbreviate *CheeseyVegetableTopping* as *CVTopping*. The unsatisfiability of concept *IceCream* does not affect that $IceCream \sqsubseteq \exists hasTopping$ and $\exists hasTopping \sqsubseteq Pizza$ can be inferred by PROSE while $\exists hasTopping \sqsubseteq Pizza$ cannot be inferred by ParOWL. *IceCream* is taken as the empty concept and then we can infer $IceCream \sqsubseteq Pizza$ and $IceCream \sqsubseteq \neg Pizza$. PROSE infers $CVTopping \sqsubseteq \neg American$ and $CVTopping \sqsubseteq \neg Pizza$ while ParOWL with internal GCIs and strong GCIs infers $CVTopping \sqsubseteq American$ and $CVTopping \sqsubseteq Pizza$. Intuitively, *CVTopping* is just a part of *Pizza*, but not a subconcept of *Pizza*. The results of PROSE are more rational than those of ParOWL since PROSE prevents transmitting the modeling error.

5.2.3. Summary

When the ontology is inconsistent, classical reasoners fail to answer any reasonable results, but both PROSE and ParOWL can still give some meaningful results under paraconsistent reasoning. As a successor of ParOWL, PROSE normally reports more results with classical truth value (T and F) and less results of unknown (U). The results of PROSE are in general more intuitive than those of ParOWL.

We also note that we did not show the running times of both systems in the table. The (strong and weak) transformations of PROSE and ParOWL are general very fast (linear time). However, it takes much more time for classical reasoners (e.g., Pellet) to answer over the transformed ontology, which is also confirmed in [24]. One possible reason is that the transformation introduce many negations and disjunctions, which makes the reasoning more difficult. Optimizing the performance is an interesting topic for future work.

6. Related works

In this section, we mainly compare QCDL with existing paraconsistent DLs.

Compared with four-valued DL [23,24], by Table 2, we find that disjunctive syllogism (DS), intuitive equivalence (IE) and resolution hold in QCDLs while they do not hold in four-valued DLs. Moreover, modus ponens (MP) and modus tollens

[2] Available at http://www.co-ode.org/ontologies/pizza/pizza.owl.

(MT) hold in QCDLs while they do not hold in four-valued DL with material inclusions and internal inclusions. Besides, the transitivity holds in four-valued DL while it does not hold in QCDLs. In this sense, the principle of tolerating inconsistency in reasoning with QCDL, where conclusions can be not allowed in next reasoning, is different from that of tolerating inconsistency in reasoning with four-valued DL, where a concept and the negation of that concept are no longer opposite to each other in all cases. In four-valued DL, there are three kinds of concept inclusions and a kind of role inclusions. Our work is integrating three kinds of inclusions (including concept inclusions and role inclusions) of four-valued DL into one kind of inclusions to make users more convenient and intuitive. More detailed comparison can be found in Table 2.

Based on four-valued DL, two kinds of three-valued DLs, namely, paradoxical DL [45] and three-valued DL [29] are presented recently. Compared with four-valued DL, they mainly handle inconsistent knowledge but do not handle incomplete knowledge. A direct advantage of them is that they satisfy the excluded middle which fails in QCDL. However, because they are based on four-valued DL, they inherit most of features of four-valued DL including some shortages (in our view). For instance, the disjunctive syllogism, intuitive equivalence and resolution fail in both of them. Moreover, interpretations defined in paradoxical DL and three-valued DL are still four-valued interpretations with some restrictions while the weak interpretation in QCDL extends four-valued interpretation in QC negation of concepts. In addition, the strong interpretation in QCDL, which enhances inference power, is difficultly to represented in both paradoxical DL and three-valued DL.

There exist some variants of four-valued DL such as $\mathcal{PALC}$ presented in [20]. $\mathcal{PALC}$ is obtained from a description logic (called $\mathcal{ALC}^n_\sim$) with such a dual (or multiple)-interpretation semantics by adding a weak negation in order to tolerate inconsistency where the weak negation is identical to the classical negation and the classical negation is identical to the QC negation in QCDL. The weak negation is used to tolerate inconsistency and the classical negation is used to implement paraconsistent reasoning. In $\mathcal{PALC}$, the satisfaction of GCIs is defined by the internal inclusion. In this sense, $\mathcal{PALC}$ can be taken as our weak semantics for DL.

Recently, [17] presents a quasi-classical semantics for DL where each quasi-classical model is a subset of Herbrand base, which is obtained by grounding all concepts and roles in a Herbrand Universe (a set of constants). In this sense, we think that the semantics could be taken as some kind of restricted version of our proposal semantics. Besides, similar to the tableau in developed in [44], the tableau calculus introduced in [17] works axioms while the tableau calculus is difficult to be technically extended in expressive DLs such as $\mathcal{SHOIN}(\mathbf{D})$.

Compared with paraconsistent approaches based on repairing [18,33,22,10] where a new consistent KB or models of KB are restored from an inconsistent KB by removing some knowledge causing inconsistency, our approach does not reject any knowledge but tolerate inconsistent knowledge in reasoning.

Similarly, those paraconsistent approaches based on argumentation presented by [12,43] introduce some partial orders (argument principles) of all consistent subsets of an inconsistent KB to select some expected consistent subsets for reasoning. Our approach adopts a totally different principle from those approaches. In addition, QCDL, including paradoxical DL, four-valued DL and $\mathcal{PALC}$, is monotonic.

As an important member of the multi-valued DL family, fuzzy description logics [40,6,5,9] can reason with uncertain knowledge in DL. Fuzzy DL admits truth values different from "true" and "false", each of which is intuitively taken as a certain degree. Usually, the set of possible truth values is the whole interval $[0, 1]$. Though some properties such as MP, MT and DS are valid in some fuzzy DL, the main difference between fuzzy logic and multi-valued logic is in the aims.

7. Conclusion

In this paper, we introduced a new description logic called quasi-classical description logic (QCDL) and investigated the properties and some important reasoning tasks of QCDL. We proved that QCDL can be used to tolerate inconsistency in reasoning with DL. We developed a transformation-based algorithm to transform QCDL to DL and reduce reasoning problems in QCDL to those in DL. Based on this algorithm, we have built a paraconsistent OWL DL prototype reasoner PROSE. The experiments showed that we have improved paraconsistent reasoning based on four-valued semantics.

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